

8.4 Integration of Rational function By Partial Fractions.

$$\text{[12]} \quad \int \frac{2x + 1}{x^2 - 7x + 12}$$

$$\int \frac{2x+1}{(x-4)(x-3)} = \frac{A}{x-4} + \frac{B}{x-3}$$

$$2x + 1 = A(x-3) + B(x-4)$$

$$\text{[x=4]} \longrightarrow A = 9$$

$$\text{[x=3]} \longrightarrow B = 7$$

$$I = \int \frac{9}{x-4} dx + \int \frac{7}{x-3} dx$$

$$= 9 \ln|x-4| + 7 \ln|x-3| + C$$

$$\boxed{14} \int_{1/2}^1 \frac{y+4}{y^2+y} dy$$

$$\frac{y+4}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

$$y+4 = A(y+1) + By$$

$$y=0 \longrightarrow \boxed{A=4}$$

$$y=-1 \longrightarrow \boxed{B=-3}$$

$$I = \int \frac{4}{y} dy + \int \frac{-3}{y+1} dy$$

$$= 4 \ln|y| - 3 \ln|y+1| + c$$

$$\int_{1/2}^1 \frac{y+4}{y^2+y} = \left[4 \ln|y| - 3 \ln|y+1| \right]_{1/2}^1$$

$$= -3 \ln 2 - 4 \ln\left(\frac{1}{2}\right) + 3 \ln\left(\frac{3}{2}\right)$$

$$= -3 \ln 2 + 4 \ln 2 + 3 \ln 3 - 3 \ln 2$$

$$= -2 \ln 2 + 3 \ln 3 = \ln \frac{27}{4}$$

$$\boxed{15} \int \frac{dt}{t^3 + t^2 - 2t}$$

$$\int \frac{dt}{t(t+2)(t-1)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1}$$

$$1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2)$$

$$\boxed{t=0} \rightarrow \boxed{A = -\frac{1}{2}}$$

$$\boxed{t=-2} \rightarrow \boxed{B = \frac{1}{6}}$$

$$\boxed{t=1} \rightarrow \boxed{C = \frac{1}{3}}$$

$$I = \int \frac{-1/2}{t} dt + \int \frac{1/6}{t+2} dt + \int \frac{1/3}{t-1} dt$$

$$= -\frac{1}{2} \ln|t| + \frac{1}{6} \ln|t+2| + \frac{1}{3} \ln|t-1| + C.$$

$$\boxed{20} \quad I = \int \frac{x^2}{(x-1)(x^2+2x+1)} dx$$

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$\boxed{x=1} \longrightarrow 4A = 1 \longrightarrow \boxed{A = \frac{1}{4}}$$

$$\boxed{x=-1} \longrightarrow -2C = 1 \longrightarrow \boxed{C = -\frac{1}{2}}$$

$$\boxed{x=0} \longrightarrow A + B + C = 0$$

$$\frac{1}{4} + B - \frac{1}{2} = 0 \longrightarrow \boxed{B = \frac{3}{4}}$$

$$I = \int \frac{1/4}{x-1} + \int \frac{3/4}{x+1} + \int \frac{-1/2}{(x+1)^2}$$

$\hookrightarrow u = x+1$

$$= \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2(1+x)} + C$$

$$\boxed{23} \quad I = \int \frac{y^2 + 2y + 1}{(y^2 + 1)^2} dy$$

$$\frac{y^2 + 2y + 1}{(y^2 + 1)^2} = \frac{Ay + B}{(y^2 + 1)} + \frac{Cy + D}{(y^2 + 1)^2}$$

$$y^2 + 2y + 1 = (Ay + B)(y^2 + 1) + Cy + D$$

$$y^2 + 2y + 1 = Ay^3 + Ay + By^2 + B + Cy + D$$

$$y^2 + 2y + 1 = Ay^3 + By^2 + (A + C)y + (B + D)$$

$$y^3: \quad \boxed{A = 0}$$

$$y^2: \quad \boxed{B = 1}$$

$$y: \quad \boxed{C = 2}$$

$$y^0: \quad \boxed{D = 0}$$

$$I = \int \frac{1}{y^2 + 1} dy + \int \frac{2y}{(y^2 + 1)^2} dy$$

$$\text{let } U = y^2 + 1 \\ dU = 2y dy$$

$$I = \tan^{-1}(y) + \int \frac{1}{U^2} dU$$

$$I = \tan^{-1}(y) + \frac{-1}{y^2 + 1} + C$$

$$= \tan^{-1}(y) - \frac{1}{y^2 + 1} + C$$

$$\boxed{29} \int \frac{x^2}{x^4 - 1} dx$$

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) \\ (x-1)(x+1)(x^2 + 1)$$

$$\frac{x^2}{x^4 - 1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$x^2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

$$\boxed{x=1} \longrightarrow 1 = 4A \longrightarrow A = \frac{1}{4}$$

$$\boxed{x=-1} \longrightarrow 1 = -4B \longrightarrow B = -\frac{1}{4}$$

$$\boxed{x=0} \longrightarrow 0 = A - B - D$$

$$0 = \frac{1}{4} - \frac{1}{4} - D \longrightarrow D = \frac{1}{2}$$

$$\boxed{x=2} \longrightarrow 4 = 15A + 5B + (2C+D)(3)$$

$$4 = \frac{15}{4} + \frac{-5}{4} + 6C + \frac{3}{2}$$

$$4 = 4 + 6C \longrightarrow \boxed{C=0}$$

$$I = \int \frac{1/4}{x-1} dx - \int \frac{1/4}{x+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx$$

$$= \frac{1}{4} \ln |x-1| - \frac{1}{4} \ln |x+1| + \frac{1}{2} \tan^{-1} x + C = \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{2} \tan^{-1} x + C$$

$$\boxed{37} \int \frac{y^4 + y^2 - 1}{y^3 + y} dy$$

$$\begin{array}{r} y \\ y^3 + y \overline{) y^4 + y^2 - 1} \\ \underline{-y^4 - y^2} \\ -1 \end{array}$$

$$\int y dy - \int \frac{1}{y^3 + y} dy$$

$$= \frac{y^2}{2} - \underbrace{\int \frac{1}{y(y^2+1)} dy}_{I_1}$$

$$= \frac{1}{2} y^2 - I_1$$

$$\text{For } I_1 = \int \frac{1}{y(y^2+1)} dy$$

$$\boxed{\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1}}$$

$$1 = A(y^2+1) + (By+C)y$$

$$1 = Ay^2 + A + By^2 + Cy$$

$$A+B=0, \quad \boxed{C=0}, \quad \boxed{A=1}, \quad \boxed{B=-1}$$

$$I_1 = \int \frac{1}{y} dy + \int \frac{-y}{y^2+1} dy = \ln|y| - \frac{1}{2} \ln|y^2+1| + C_1$$

$$\boxed{I = \frac{1}{2} y^2 - \ln|y| + \frac{1}{2} \ln(y^2+1) + C}$$

$$\boxed{42} \quad I = \int \frac{\sin \theta}{\cos^2 \theta + \cos \theta - 2} d\theta$$

$$\text{let } y = \cos \theta \\ dy = -\sin \theta d\theta$$

$$I = \int \frac{-dy}{y^2 + y - 2}$$

$$= - \int \frac{1}{(y+2)(y-1)} dy$$

$$\boxed{\frac{1}{(y+2)(y-1)} = \frac{A}{y+2} + \frac{B}{y-1}}$$

$$1 = A(y-1) + B(y+2)$$

$$\boxed{y=-2} \longrightarrow A = -\frac{1}{3}$$

$$\boxed{y=1} \longrightarrow B = \frac{1}{3}$$

$$I = - \int \frac{-1/3}{y+2} dy - \int \frac{1/3}{y-1} dy$$

$$= \frac{1}{3} \int \frac{1}{y+2} dy - \frac{1}{3} \int \frac{1}{y-1} dy$$

$$= \frac{1}{3} \ln |y+2| - \frac{1}{3} \ln |y-1| + C$$

$$= \frac{1}{3} \ln \left| \frac{y+2}{y-1} \right| + C = \frac{1}{3} \ln \left| \frac{\cos \theta + 2}{\cos \theta - 1} \right| + C$$

47

$$I = \int \frac{\sqrt{x+1}}{x} dx$$

$$\text{let } y = \sqrt{x+1} \rightarrow x+1 = y^2$$

$$dx = 2y dy$$

$$I = \int \frac{y}{y^2-1} 2y dy$$

$$= \int \frac{2y^2}{y^2-1} dy$$

$$\begin{array}{r} 2 \\ y^2-1 \overline{) 2y^2} \\ \underline{-2y^2 + 2} \\ 2 \end{array}$$

$$= \int \left(2 + \frac{2}{y^2-1} \right) dy$$

$$= \int 2 dy + 2 \int \frac{1}{(y-1)(y+1)} dy$$

$$= 2y + \frac{2}{2} \int \frac{1}{y-1} - \frac{2}{2} \int \frac{1}{y+1}$$

$$= 2y + \ln|y-1| - \ln|y+1| + C$$

$$\left\{ \begin{array}{l} \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} \\ \boxed{y=1} \rightarrow A = \frac{1}{2} \\ \boxed{y=-1} \rightarrow B = -\frac{1}{2} \end{array} \right.$$

$$= 2y + \ln \left| \frac{y-1}{y+1} \right| + C$$

$$I = 2\sqrt{x+1} + \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C$$

$$\boxed{49} \quad I = \int \frac{1}{x(x^4+1)} dx$$

$$= \int \frac{1}{y \cdot x} \frac{dy}{4x^3}$$

$$= \frac{1}{4} \int \frac{1}{y(y-1)} dy$$

$$\boxed{\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}}$$

$$I = \frac{1}{4} \int \frac{-1}{y} + \frac{1}{4} \int \frac{1}{y-1}$$

$$1 = A(y-1) + B(y)$$

$$\boxed{y=0} \longrightarrow \boxed{A=-1}$$

$$\boxed{y=1} \longrightarrow \boxed{B=1}$$

$$I = -\frac{1}{4} \ln |y| + \frac{1}{4} \ln |y-1| + C$$

$$I = \frac{1}{4} \ln \left| \frac{y-1}{y} \right| + C$$

$$I = \frac{1}{4} \ln \left(\frac{x^4}{x^4+1} \right) + C$$

54

Solve the initial value problem (I.V.P)

$$(t+1) \frac{dx}{dt} = x^2 + 1, \quad t > -1, \quad x(0) = 0$$

Sol:-

$$(t+1) dx = (x^2 + 1) dt$$

$$\int \frac{dx}{x^2 + 1} = \int \frac{dt}{t+1}$$

$$\tan^{-1} x + C_1 = \ln |t+1| + C_2$$

$$\tan^{-1} x = \ln |t+1| + C$$

$$x(0) = 0 \longrightarrow x = 0, \quad t = 0$$

$$\tan^{-1}(0) = \ln 1 + C$$

$$0 = 0 + C \longrightarrow C = 0$$

$$\tan^{-1} x = \ln(t+1), \quad t > -1$$

$$x = \tan(\ln(t+1))$$