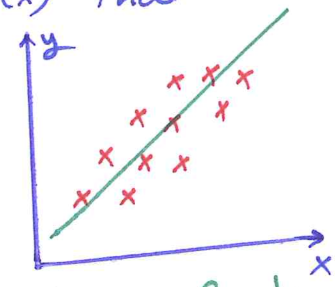
Ch5 : Curve fitting

- Given a distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
- Can we find a formula (or a curve) $y = f(x)$ that fits (or relates) these points?



- There are many different possibilities for the type of function that can be used.
- In this section we will study the class of linear function of the form: $y = f(x) = Ax + B$
- In ch4, we saw how to construct a polynomial that passes through a set of points. However, since f needs not to pass through these point, we can not use interpolation.
- Now we need to find A and B so we minimize the error (or deviation or residual): $|e_k| = |f(x_k) - y_k|$, $k = 1, 2, \ldots, n$
- To handle the errors, we use norms to measure how far the curve $y = f(x)$ lies from the data.
- we consider the following norms:

[1] Maximum Error (ME): $E_\infty(f) = \text{Max}\, |e_k|$, $k \in \{1, 2, \ldots, n\}$

[2] Average Error (AVE): $E_1(f) = \frac{1}{n} \sum_{k=1}^{n} |e_k|$

[3] Root-Mean-Square Error (RMSE): $E_2(f) = \sqrt{\frac{1}{n} \sum_{k=1}^{n} |e_k|^2}$

Exp Compare the ME, AVE, and the RMSE for the linear approximation $y = f(x) = 2x + 1$ to the data points: $(1, 1.9)$, $(-1, -0.7)$, $(0, 1.2)$.

x_k	y_k	$f(x_k) = 2x_k + 1$	$\lvert e_k \rvert = \lvert f(x_k) - y_k \rvert$	$\lvert e_k \rvert^2$
1	1.9	3	1.1	1.21
-1	-0.7	-1	0.3	0.09
0	1.2	1	0.2	0.04

$$E_\infty(f) = \operatorname{Max}\{1.1, 0.3, 0.2\} = 1.1$$

$$E_1(f) = \frac{1.1 + 0.3 + 0.2}{3} = \frac{1.6}{3} = 0.5\bar{3}$$

$$E_2(f) = \sqrt{\frac{1.21 + 0.09 + 0.04}{3}} = \sqrt{\frac{1.34}{3}} = \sqrt{0.44\bar{6}} \simeq 0.67$$

- Note that
  - E_∞ is the largest and if one point is badly in the error, its value determines E_∞.
  - E_1 averages the abs. value of the error. It is often used because it is easy to compute.
  - E_2 is the traditional choice because it is much easier to minimize.

Our task is to find a curve fitting that minimize E_2.

Finding the Least-Squares Line

- Given n distinct points : $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- The least-squares line $y = f(x) = Ax + B$ is the line that minimizes the RMSE $E_2(f)$:

$$E_2(f) = \sqrt{\frac{\sum (f(x_i) - y_i)^2}{n}} \Rightarrow n E_2^2 = \sum_{i=1}^{n} (f(x_i) - y_i)^2$$

- E_2 is minimized iff $E(A,B) = \sum_{i=1}^{n} (Ax_i + B - y_i)^2$ is minimized

$$\frac{\partial E}{\partial A} = 2 \sum_{i=1}^{n} (Ax_i + B - y_i) x_i = 0 \Leftrightarrow \sum_{i=1}^{n} (Ax_i^2 + Bx_i - y_i x_i) = 0$$

$$A \sum_{i=1}^{n} x_i^2 + B \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \quad \cdots \quad (1)$$

$$\frac{\partial E}{\partial B} = 2 \sum_{i=1}^{n} (Ax_i + B - y_i) = 0 \Leftrightarrow \sum_{i=1}^{n} (Ax_i + B - y_i) = 0$$

$$A \sum_{i=1}^{n} x_i + nB = \sum_{i=1}^{n} y_i \quad \cdots \quad (2)$$

- Equations (1) and (2) are called the linear normal equations and used to find the coefficients A and B.

Exp Find the least-squares line for the following data points: (1,2), (3,-1), (2,-1), (0,1), (-1,3)

x	y	xy	x^2
1	2	2	1
3	-1	-3	9
2	-1	-2	4
0	1	0	0
-1	3	-3	1
5	4	-6	15

Normal Equations:

$$A\sum_{i=1}^{5} x_i^2 + B\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} x_i y_i$$

$$\boxed{15A + 5B = -6} \quad (1)$$

$$A\sum_{i=1}^{5} x_i + nB = \sum_{i=1}^{5} y_i$$

$$\boxed{5A + 5B = 4} \quad (2)$$

- $$A = \frac{\begin{vmatrix} -6 & 5 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 15 & 5 \\ 5 & 5 \end{vmatrix}} = \frac{-50}{50} = -1$$

$$B = \frac{\begin{vmatrix} 15 & -6 \\ 5 & 4 \end{vmatrix}}{50} = \frac{90}{50} = 1.8$$

Hence, $y = Ax + B$
$= -x + 1.8$

Exp Find the normal equation for the best fit of the form $y = Ax^m$ where m is known constant.

- $$E(A) = \sum_{i=1}^{n} \left(f(x_i) - y_i \right)^2 = \sum_{i=1}^{n} \left(A x_i^m - y_i \right)^2$$

- $$E'(A) = 2\sum_{i=1}^{n} \left(A x_i^m - y_i \right) x_i^m = 0 \iff$$

$$\sum_{i=1}^{n} A x_i^{2m} - \sum_{i=1}^{m} x_i^m y_i = 0 \iff$$

$$\boxed{A = \frac{\sum_{i=1}^{n} x_i^m y_i}{\sum_{i=1}^{n} x_i^{2m}}}$$

Exp Find the power fits $y = Ax^2$ for the following data. Then find $E_2(f)$. 111

i	x_i	y_i	x_i^2	$x_i^2 y_i$	x_i^4	$f(x_i)$	$\lvert e_i\rvert^2 = \lvert f(x_i) - y_i\rvert^2$
1	2.0	5.1	4	20.4	16	6.748	2.715904
2	2.3	7.5	5.29	39.675	27.9841	8.92423	2.0284310929
3	2.6	10.6	6.76	71.656	45.6976	11.40412	0.6466089744
4	2.9	14.4	8.41	121.104	70.7281	14.18767	0.0450840289
5	3.2	19.0	10.24	194.56	104.8576	17.27488	2.9760390144
				447.395	265.2674		8.4120671106

- $$A = \frac{\sum_{i=1}^{5} x_i^2 y_i}{\sum_{i=1}^{5} x_i^4} = \frac{447.395}{265.2674} \approx 1.687$$

Hence, $y = f(x) = Ax^2 = 1.687x^2$

- $$E_2(f) = \sqrt{\frac{\sum_{i=1}^{5} \lvert e_i\rvert^2}{5}} = \sqrt{\frac{8.4120671106}{5}} = \sqrt{1.68241342221}$$
$$= 1.2970788034$$

Exp Given the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.
Find the normal equations for the best fit of the form $y = f(x) = Ax^2 + Bx + C$

- $E(A,B,C) = \sum_{i=1}^{n} \left(A x_i^2 + B x_i + C - y_i \right)^2$

$$\frac{\partial E}{\partial A} = 2 \sum_{i=1}^{n} \left(A x_i^2 + B x_i + C - y_i \right) x_i^2 = 0 \iff$$

$$\boxed{\left(\sum_{i=1}^{n} x_i^4 \right) A + \left(\sum_{i=1}^{n} x_i^3 \right) B + \left(\sum_{i=1}^{n} x_i^2 \right) C = \sum_{i=1}^{n} y_i x_i^2} \quad (1)$$

$$\frac{\partial E}{\partial B} = 2 \sum_{i=1}^{n} \left(A x_i^2 + B x_i + C - y_i \right) x_i = 0 \iff$$

$$\boxed{\left(\sum_{i=1}^{n} x_i^3 \right) A + \left(\sum_{i=1}^{n} x_i^2 \right) B + \left(\sum_{i=1}^{n} x_i \right) C = \sum_{i=1}^{n} y_i x_i} \quad (2)$$

$$\frac{\partial E}{\partial C} = 2 \sum_{i=1}^{n} \left(A x_i^2 + B x_i + C - y_i \right) = 0 \iff$$

$$\boxed{\left(\sum_{i=1}^{n} x_i^2 \right) A + \left(\sum_{i=1}^{n} x_i \right) B + nC = \sum_{i=1}^{n} y_i} \quad (3)$$

- To find A, B, C we solve the three equations above.

Exp Find the least-squares parabola for the four points $(-3,3)$, $(0,1)$, $(2,1)$ and $(4,3)$.

x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$	
-3	3	9	-27	81	-9	27	
0	1	0	0	0	0	0	
2	1	4	8	16	2	4	
4	3	16	64	256	12	48	
3	8	29	45	353	5	79	Total

- The least-squares parabola is $y = f(x) = Ax^2 + Bx + C$
- To find A, B, C we use equations ①, ②, ③ in page 112:

$$\left.\begin{aligned} 353A + 45B + 29C &= 79 \\ 45A + 29B + 3C &= 5 \\ 29A + 3B + 4C &= 8 \end{aligned}\right\} \Rightarrow \quad A = \frac{585}{3278} = 0.178462$$

$$B = -0.192495$$

$$C = 0.850519$$

Hence, $y = f(x) = 0.178462\, x^2 - 0.192495\, x + 0.850519$

Exp Find the best fit of the form $y = A\sin(\pi x)$

$$E(A) = \sum_{i=1}^{n} \left(A\sin(\pi x_i) - y_i\right)^2$$

$$E'(A) = 2\sum_{i=1}^{n} \left(A\sin(\pi x_i) - y_i\right)\sin(\pi x_i) = 0$$

$$A = \frac{\sum_{i=1}^{n} y_i \sin(\pi x_i)}{\sum_{i=1}^{n} \sin^2(\pi x_i)}$$

Linearization

Exp • Given the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

• Find the least-squar exponential curve of the form

$$y = C e^{Dx}$$

• $E(C, D) = \sum_{i=1}^{n} \left(C e^{Dx_i} - y_i\right)^2$

• $\frac{\partial E}{\partial C} = 2 \sum_{i=1}^{n} \left(C e^{Dx_i} - y_i\right) e^{Dx_i} = 0 \quad \Leftrightarrow$

$$\boxed{C \sum_{i=1}^{n} e^{2Dx_i} - \sum_{i=1}^{n} y_i e^{Dx_i} = 0} \text{ — } ①$$

• $\frac{\partial E}{\partial D} = 2 \sum_{i=1}^{n} \left(C e^{Dx_i} - y_i\right) \cancel{C} x_i e^{Dx_i} = 0 \quad \Leftrightarrow$

$$\boxed{C \sum_{i=1}^{n} x_i e^{2Dx_i} - \sum_{i=1}^{n} y_i x_i e^{Dx_i} = 0} \text{ — } ②$$

• The normal equations ① and ② are hard to solve and find C and D.

• So we use a technique called linearization.

- linearization for $y = c\,e^{Dx}$ works like this:
  - Take logarithm of both sides:

$$\ln y = Dx + \ln c$$

  - Then introduce the change of variables:

$$Y = Dx + E \quad \text{where } Y = \ln y, \quad E = \ln C$$

  - Now use the linear normal equations page 109

$$D\sum_{i=1}^{n} x_i^2 + E\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i Y_i$$

$$D\sum_{i=1}^{n} x_i + nE = \sum_{i=1}^{n} Y_i \qquad *$$

  - Solve these equations for D and E $\Rightarrow$

Then $C = e^{E}$ and so $y = f(x) = C\,e^{Dx}$

Exp Find the exponential fit $y = C\,e^{Dx}$ using linearization for the following five data points:

$(0, 1.5), (1, 2.5), (2, 3.5), (3, 5), (4, 7.5)$

- First we solve the linear normal equations $*$ and find the constants D and E
- Then we find $C = e^{E}$
- Hence, $y = f(x) = C\,e^{Dx}$

x_i	y_i	x_i^2	$Y_i = \ln y_i$	$x_i Y_i$
0	1.5	0	0.405465	0
1	2.5	1	0.916291	0.916291
2	3.5	4	1.252763	2.505526
3	5	9	1.609438	4.828314
4	7.5	16	2.014903	8.059612
Total		30	6.198860	16.309743

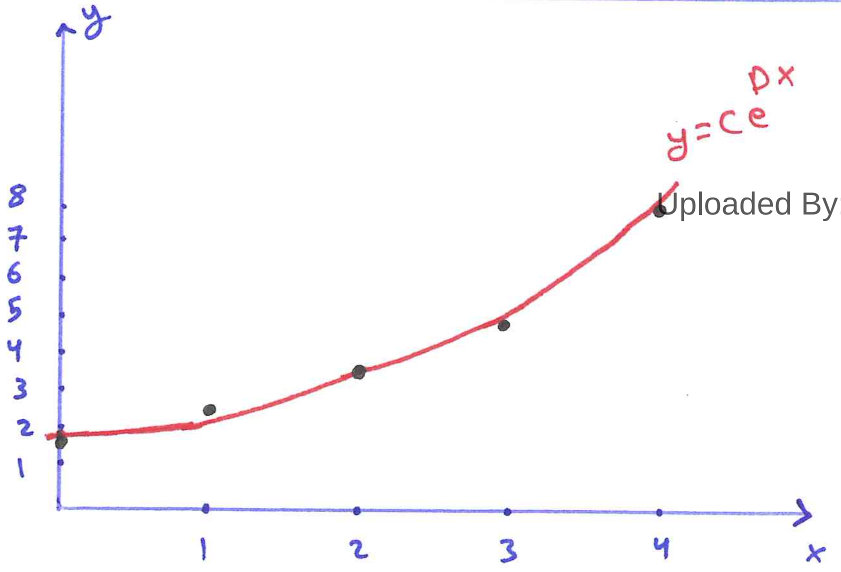
- The linear normal equations become:

$$30\,D + 10\,E = 16.309743$$
$$10\,D + 5\,E = 6.198860$$

- The solution is $D = 0.3912023$ and $E = 0.457367$
- Now we find $C = e^{E} = e^{0.457367} = 1.579910$
- Hence, $y = f(x) = C\,e^{Dx}$

$$= 1.579910\, e^{0.3912023\,x}$$

The exponential fit $y = 1.579910\,e^{0.3912023x}$ obtained by using the linearization method.



Exp Given the following data

x	1	2	4	5
y	2	8	4	6

Use two different linearization to find the fit of the form $g(x) = \frac{cx}{D+x}$. Then estimate y when $x = 3$.

1^{st} linearization $\quad y = \frac{cx}{D+x} \Rightarrow \frac{1}{y} = \frac{D+x}{cx} = \frac{D}{c}\frac{1}{x} + \frac{1}{c}$

• Let $Y = \frac{1}{y}$, $X = \frac{1}{x}$ $\qquad \boxed{Y = \alpha X + \beta}$

• Now solve normal equations: $\qquad$ where $\alpha = \frac{D}{c}$, $\beta = \frac{1}{c}$

$\alpha \sum X_i^2 + \beta \sum X_i = \sum X_i Y_i \Rightarrow 1.3525\alpha + 1.95\beta = 0.6584$

$\alpha \sum X_i + n\beta = \sum Y_i \Rightarrow 1.95\alpha + 4\beta = 1.0417$

$\Downarrow$

x	y	$X_i = \frac{1}{x_i}$	$Y_i = \frac{1}{y_i}$	X_i^2	$X_i Y_i$
1	2	1	0.5	1	0.5
2	8	0.5	0.125	0.25	0.0625
4	4	0.25	0.25	0.0625	0.0625
5	6	0.2	0.1667	0.04	0.0334
		1.95	1.0417	1.3525	0.6584

$\alpha = 0.3767$

$\beta = 0.07777$

$c = \frac{1}{\beta} = 12.86$

$D = \alpha c = 4.844$

$$g(x) = \frac{cx}{D+x} = \frac{12.86x}{4.844 + x}$$

when $x = 3 \Rightarrow y(3) \approx g(3) = \frac{(12.86)(3)}{4.844+3} \approx 4.918$

2^{nd} linearization

$$y = \frac{Cx}{D+x} \Rightarrow \frac{y}{x} = \frac{C}{D+x} \Rightarrow \frac{x}{y} = \frac{D+x}{C}$$

$$\frac{x}{y} = \frac{D}{C} + \frac{1}{C}x$$

- Let $Y = \frac{x}{y}$, $A = \frac{1}{C}$, $B = \frac{D}{C}$, $\boxed{Y = Ax + B}$
- Now solve the normal equations:

$$A \sum x_i^2 + B \sum x_i = \sum x_i Y_i \Rightarrow 46A + 12B = 9.165$$
$$A \sum x_i + Bn = \sum Y_i \Rightarrow 12A + 4B = 2.583$$

$$\Downarrow$$

x	y	$Y_i = \frac{x_i}{y_i}$	x_i^2	$x_i Y_i$
1	2	0.5	1	0.5
2	8	0.25	4	0.5
4	4	1	16	4
5	6	0.833	25	4.165
12		2.583	46	9.165

$A = 0.1416$
$B = 0.221$

$C = \frac{1}{A} = 7.06$

$D = BC = 1.56$

$$g(x) = \frac{Cx}{D+x} = \frac{7.06x}{1.56+x}$$

when $x = 3 \Rightarrow y(3) \approx g(3) = \frac{(7.06)(3)}{1.56+3} = 4.645$

Exp Given the data $(0,1), (1,2), (3,4), (5,3)$. Use linearization to find the best fitting curve of the form $y = A x^{\beta}$ through these points.

- $y = A x^{\beta} \Rightarrow \ln y = \ln(A x^{\beta}) = \ln A + \beta \ln x$
- Let $\bar{Y} = \ln y$ and $\bar{X} = \ln x$ and $\alpha = \ln A$

$$\bar{Y} = \alpha + \beta \bar{X}$$

- The normal linear equations are:

$$\beta \sum \bar{X}_i^2 + \alpha \sum \bar{X}_i = \sum \bar{X}_i \bar{Y}_i$$

$$\beta \sum \bar{X}_i + \alpha n = \sum \bar{Y}_i$$

x_i	y_i	$\bar{Y}_i = \ln y_i$	$\bar{X}_i = \ln x_i$	$\bar{X}_i^2$	$\bar{X}_i \bar{Y}_i$
0	1		undefined		
1	2	0.6931	0	0	0
3	4	1.386	1.099	1.208	1.523
5	3	1.099	1.609	2.589	1.768
		3.178	2.708	3.797	3.291

→ we ignore the point $(0,1)$ $\Downarrow$ $n = 3$

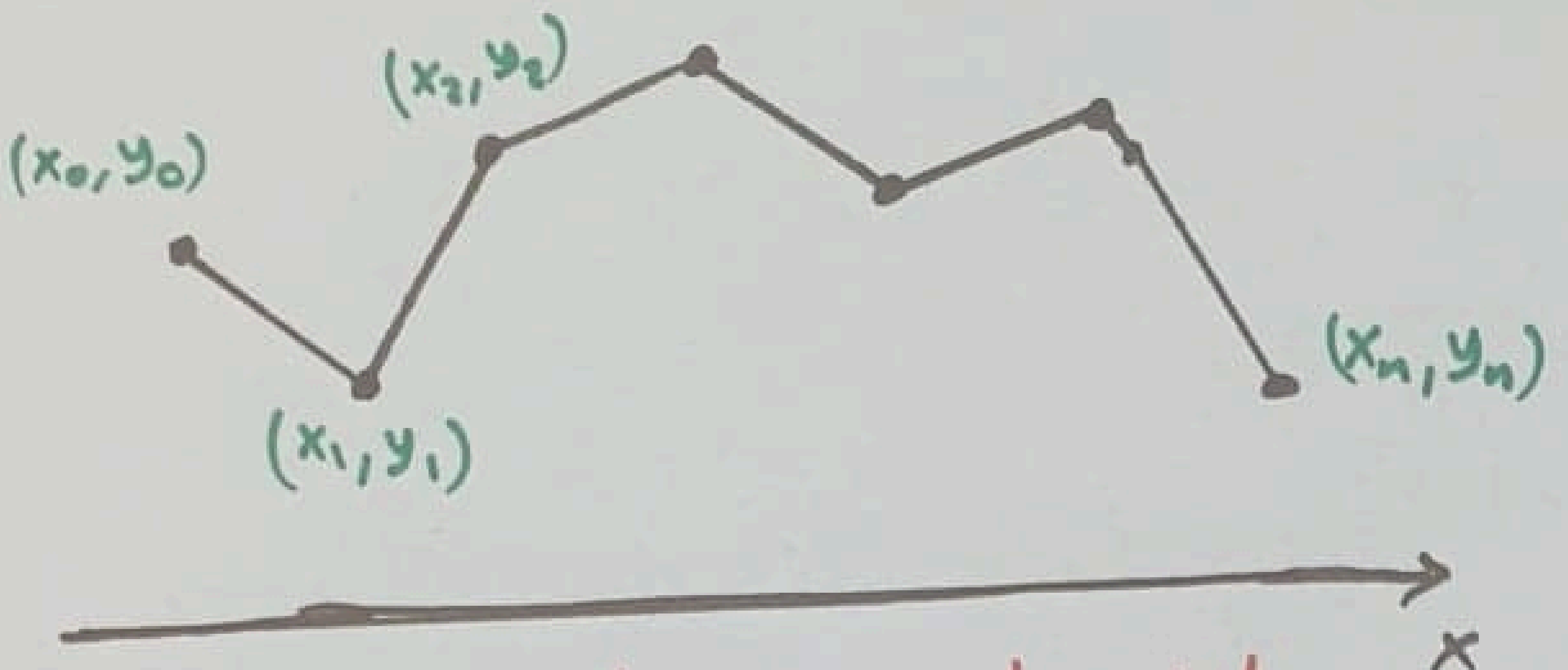
$$\left.\begin{array}{l} 3.797\,\beta + 2.708\,\alpha = 3.291 \\ 2.708\,\beta + 3\,\alpha = 3.178 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha = 0.7775 \\ \beta = 0.3122 \end{array}$$

But $\alpha = \ln A \Rightarrow A = e^{\alpha} = (2.178)^{0.7775} = 1.832$

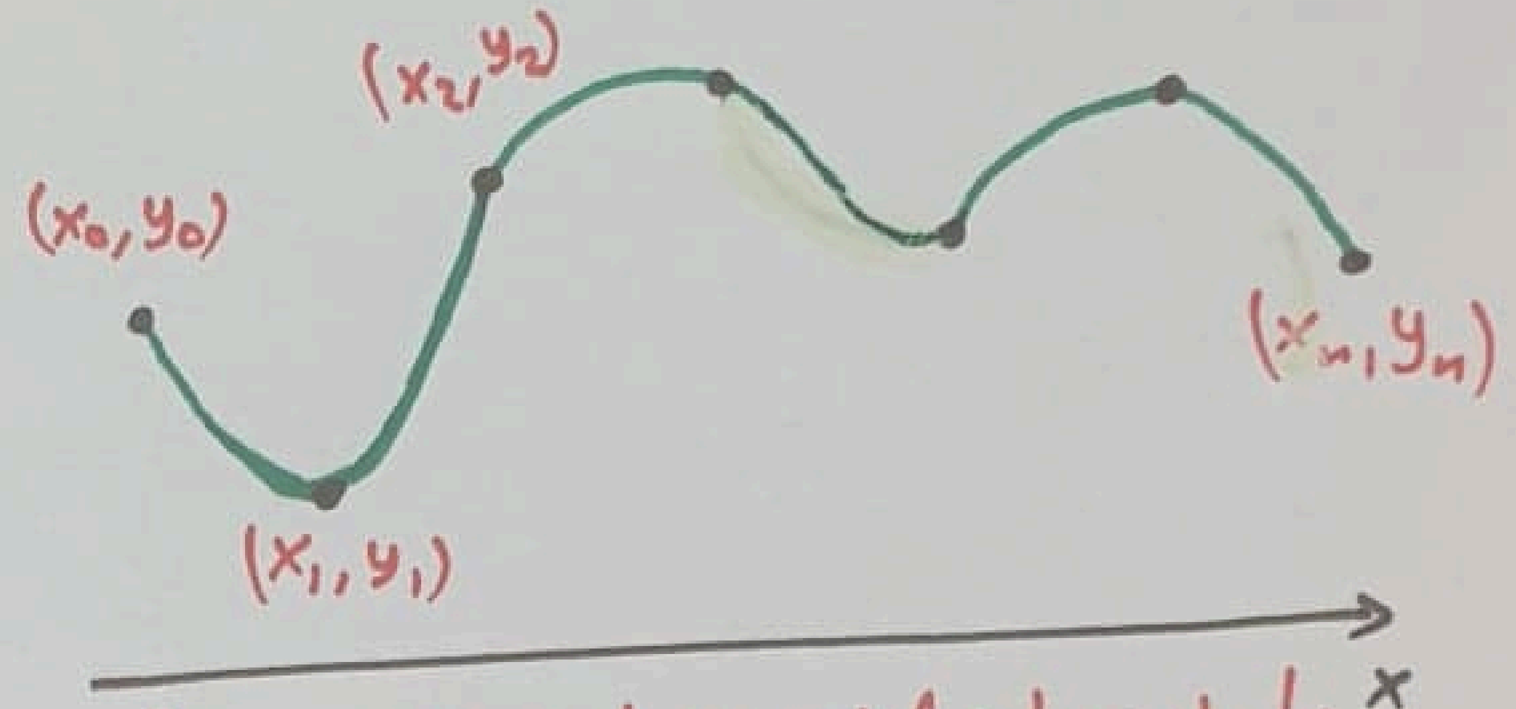
Hence, $y = A x^{\beta} = 1.832\, x^{0.3122}$

5.3 Interpolation by Spline Functions

- In this section we study a piecewise interpolation.
- Piecewise interpolation can be linear or nonlinear "polynomial" interpolation.



Piecewise linear interpolation
"linear spline"



Piecewise polynomial interpolation
"cubic Spline"

Def (Piecewise linear spline)

- The piecewise linear curve defined on $[x_k, x_{k+1}]$ is

$$S_k(x) = y_k + d_k(x - x_k)$$

where $k = 0, 1, 2, \dots, n-1$ and $d_k = \dfrac{y_{k+1} - y_k}{x_{k+1} - x_k}$.

- That is :

$$\begin{cases} S_0(x) = y_0 + d_0(x - x_0) & , \quad x_0 \le x \le x_1 \\ S_1(x) = y_1 + d_1(x - x_1) & , \quad x_1 \le x \le x_2 \\ \vdots & \quad \vdots \\ S_k(x) = y_k + d_k(x - x_k) & , \quad x_k \le x \le x_{k+1} \\ \vdots & \quad \vdots \\ S_{n-1}(x) = y_{n-1} + d_{n-1}(x - x_{n-1}) & , \quad x_{n-1} \le x \le x_n \end{cases}$$

Remark : The Lagrange polynomial is used to represent this piecewise linear spline : $S_k(x) = y_k \dfrac{x - x_{k+1}}{x_k - x_{k+1}} + y_{k+1} \dfrac{x - x_k}{x_{k+1} - x_k}$ for $x_k \le x \le x_{k+1}$

where $k = 0, 1, 2, \dots, n-1$

Def (Piecewise Cubic Splines)

- Given $n+1$ points: $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
- The function $S(x)$ defined by n formula on $[a,b] = [x_0, x_n]$:

$$S(x) = \begin{cases} S_0(x) = A_0(x-x_0)^3 + B_0(x-x_0)^2 + C_0(x-x_0) + D_0, & x_0 \le x \le x_1 \\ S_1(x) = A_1(x-x_1)^3 + B_1(x-x_1)^2 + C_1(x-x_1) + D_1, & x_1 \le x \le x_2 \\ \vdots & \vdots \\ S_{n-1}(x) = A_{n-1}(x-x_{n-1})^3 + B_{n-1}(x-x_{n-1})^2 + C_{n-1}(x-x_{n-1}) + D_{n-1}, & x_{n-1} \le x \le x_n \end{cases}$$

is called cubic spline iff the following conditions hold:

$\boxed{1}$
$S_0(x_0) = y_0$
$S_1(x_1) = y_1$
$S_2(x_2) = y_2$
$\vdots$
$S_{n-1}(x_{n-1}) = y_{n-1}$
$S_{n-1}(x_n) = y_n$

$n+1$ conditions (equations)

$\boxed{2}$
$S_0(x_1) = S_1(x_1)$
$S_1(x_2) = S_2(x_2)$
$\vdots$
$S_{n-2}(x_{n-1}) = S_{n-1}(x_{n-1})$

$n-1$ conditions

$\boxed{3}$
$S_0'(x_1) = S_1'(x_1)$
$S_1'(x_2) = S_2'(x_2)$
$\vdots$
$S_{n-2}'(x_{n-1}) = S_{n-1}'(x_{n-1})$

$n-1$ conditions

$\boxed{4}$
$S_0''(x_1) = S_1''(x_1)$
$S_1''(x_2) = S_2''(x_2)$
$\vdots$
$S_{n-2}''(x_{n-1}) = S_{n-1}''(x_{n-1})$

$n-1$ conditions

Remark: we use cubic splines to estimate $f(x)$ on $[a,b] = [x_0, x_n]$

1. Cubic splines produces $4n-2$ equations but we have $4n$ unknows so there is two degree of freedom (2 missing conditions).
2. We use cubic splines to estimate $f(x)$ because we can make its first and second derivatives all continuous on the large interval $[x_0, x_n]$ so that $S(x) = y$ has no sharp corners.

* Depending on the remaining two conditions, there are two types of cubic spline :

1. Clamped Cubic Spline : $S'(a) = S_0'(x_0) = f'(x_0)$
   $S'(b) = S_{n-1}'(x_n) = f'(x_n)$

2. Natural Cubic Spline : $S''(a) = S_0''(x_0) = 0$
   $S''(b) = S_{n-1}''(x_n) = 0$

Exp Given (x_0, y_0) and (x_1, y_1). Write the form of the cubic spline $S(x)$ that estimat $y = f(x)$.

$n=1$

$$S(x) = S_0(x) = A_0(x - x_0)^3 + B_0(x-x_0)^2 + C_0(x-x_0) + D_0, \quad x_0 \le x \le x_1$$

since $n = 1$

Exp Given the following data points: $\overset{x_0}{(1,2)}, \overset{x_1}{(2,3)}, \overset{x_2}{(3,5)}$. $n=2$

i. Find the natural cubic spline through these data.

ii. Find the clamped cubic spline through these data given that $f'(1) = 2$ and $f'(3) = 1$

• $n = 2 \Rightarrow$

$$S(x) = \begin{cases} S_0(x) = A_0(x-1)^3 + B_0(x-1)^2 + C_0(x-1) + D_0, & 1 \le x \le 2 \\ S_1(x) = A_1(x-2)^3 + B_1(x-2)^2 + C_1(x-2) + D_1, & 2 \le x \le 3 \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = 3A_0(x-1)^2 + 2B_0(x-1) + C_0 & , \quad 1 \le x \le 2 \\ S_1'(x) = 3A_1(x-2)^2 + 2B_1(x-2) + C_1 & , \quad 2 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6A_0(x-1) + 2B_0 & , \quad 1 \le x \le 2 \\ S_1''(x) = 6A_1(x-2) + 2B_1 & , \quad 2 \le x \le 3 \end{cases}$$

$\boxed{1} \Rightarrow \quad S_0(x_0) = y_0 \iff S_0(1) = 2 \iff \boxed{D_0 = 2}$

$S_1(x_1) = y_1 \iff S_1(2) = 3 \iff \boxed{D_1 = 3}$

$S_1(x_2) = y_2 \iff S_1(3) = 5 \iff A_1 + B_1 + C_1 + D_1 = 5$

$\iff \boxed{A_1 + B_1 + C_1 = 2} - *^1$

$\boxed{2} \Rightarrow \quad S_0(x_1) = S_1(x_1) \iff S_0(2) = S_1(2) \iff A_0 + B_0 + C_0 + D_0 = D_1$

$\iff \boxed{A_0 + B_0 + C_0 = 1} - *^2$

$\boxed{3} \Rightarrow \quad S_0'(x_1) = S_1'(x_1) \iff S_0'(2) = S_1'(2) \iff \boxed{3A_0 + 2B_0 + C_0 = C_1} - *^3$

$\boxed{4} \Rightarrow \quad S_0''(x_1) = S_1''(x_1) \iff S_0''(2) = S_1''(2) \iff \boxed{6A_0 + 2B_0 = 2B_1} - *^4$

$\boxed{1}$ For Natural Cubic spline $\Rightarrow$

$S''(a) = S_0''(x_0) = S_0''(1) = 0 \iff 2B_0 = 0 \iff \boxed{B_0 = 0}$

$S''(b) = S_1''(x_2) = S_1''(3) = 0 \iff 6A_1 + 2B_1 = 0 \iff \boxed{3A_1 + B_1 = 0} - *^5$

$*^2 \Rightarrow C_0 = 1 - A_0$ so $*^3$ becomes $3A_0 + 1 - A_0 = C_1$

$2A_0 + 1 = C_1$

$*^4 \Rightarrow B_1 = 3A_0$ so $*^1$ becomes $A_1 + 3A_0 + 2A_0 + 1 = 2$

$$\left.\begin{array}{l} A_1 + 5A_0 = 1 \\ A_1 + A_0 = 0 \end{array}\right\}$$

$*^5$ becomes $A_1 + A_0 = 0$

$A_0 = \frac{1}{4}$

$A_1 = -\frac{1}{4}$

$C_0 = 1 - A_0 = \frac{3}{4}$

$C_1 = 2A_0 + 1 = \frac{3}{2}$

$B_1 = 3A_0 = \frac{3}{4}$

Hence, the natural cubic spline is

$$s(x) = \begin{cases} s_0(x) = \frac{1}{4}(x-1)^3 + \frac{3}{4}(x-1) + 2, & 1 \leq x \leq 2 \\ s_1(x) = -\frac{1}{4}(x-2)^3 + \frac{3}{4}(x-2)^2 + \frac{3}{2}(x-2) + 3, & 2 \leq x \leq 3 \end{cases}$$

$\boxed{ii}$ For clamped cubic spline $\Rightarrow$

- $s'(a) = s_0'(x_0) = s_0'(1) = f'(1) = 2 \Leftrightarrow s_0'(1) = 2$
  $\Leftrightarrow \boxed{C_0 = 2}$

- $s'(b) = s_1'(x_2) = s_1'(3) = f'(3) = 1 \Leftrightarrow s_1'(3) = 1$
  $\Leftrightarrow \boxed{3A_1 + 2B_1 + C_1 = 1}$ $*6$

- substitute $C_0 = 2$ in $*^1, *^2, *^3, *^4$ and add $*6 \Rightarrow$

$$\left.\begin{array}{ll} A_1 + B_1 + C_1 = 2 & \cdots *^1 \\ A_0 + B_0 + C_0 = 1 & \cdots *^2 \\ 3A_0 + 2B_0 + C_0 - C_1 = 0 & \cdots *^3 \\ 6A_0 + 2B_0 - 2B_1 = 0 & \cdots *^4 \end{array}\right\} \Rightarrow \begin{array}{l} A_1 + B_1 + C_1 = 2 \\ A_0 + B_0 = -1 \\ 3A_0 + 2B_0 - C_1 = -2 \\ 6A_0 + 2B_0 - 2B_1 = 0 \\ 3A_1 + 2B_1 + C_1 = 1 \end{array}$$

- Write this system using matrix form with order A_0, B_0, A_1, B_1, C_1

$$\begin{array}{cccccc} A_0 & B_0 & A_1 & B_1 & C_1 & b \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 3 & 2 & 0 & 0 & -1 & -2 \\ 6 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 3 & 2 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 6 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ \\ R_4 - 6R_1 \\ \\ \end{array}$$

pivot

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & -4 & 0 & -2 & 0 & 6 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 4R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 & 4 & 2 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ R_4/-2 \\ R_5 - 3R_3 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 & -5 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ R_5 + R_4 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & -4 & -6 \end{array}\right]$$

$-4C_1 = -6 \Rightarrow \boxed{C_1 = \frac{3}{2}}$

$B_1 - \cancel{2}\left(\frac{3}{\cancel{2}}\right) = -1$

$\boxed{B_1 = 2}$

$A_1 + (\cancel{2}) + \left(\frac{3}{2}\right) = \cancel{2} \Rightarrow \boxed{A_1 = \frac{-3}{2}}$, $B_0 + \frac{3}{2} = -1 \Rightarrow B_0 = \frac{-5}{2}$

$A_0 + \left(\frac{-5}{2}\right) = -1 \Rightarrow A_0 = \frac{3}{2}$

Hence, the clamped cubic spline is

$$s(x) = \begin{cases} S_0(x) = \frac{3}{2}(x-1)^3 - \frac{5}{2}(x-1)^2 + 2(x-1) + 2, & 1 \le x \le 2 \\ S_1(x) = -\frac{3}{2}(x-2)^3 + 2(x-2)^2 + \frac{3}{2}(x-2) + 3, & 2 \le x \le 3 \end{cases}$$

Exp • Given $\overset{x_0\ y_0}{(0,1)}, \overset{x_1\ y_1}{(1,2)}, \overset{x_2\ y_2}{(3,4)}$

• Find the Natural Cubic Spline

• $n=2 \Rightarrow$ The cubic spline is

$$S(x) = \begin{cases} S_0(x) = A_0 x^3 + B_0 x^2 + C_0 x + D_0 , & 0 \le x \le 1 \\ S_1(x) = A_1 (x-1)^3 + B_1 (x-1)^2 + C_1 (x-1) + D_1 , & 1 \le x \le 3 \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = 3A_0 x^2 + 2B_0 x + C_0 , & 0 \le x \le 1 \\ S_1'(x) = 3A_1 (x-1)^2 + 2B_1 (x-1) + C_1 , & 1 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6A_0 x + 2B_0 , & 0 \le x \le 1 \\ S_1''(x) = 6A_1 (x-1) + 2B_1 , & 1 \le x \le 3 \end{cases}$$

$\boxed{1}$ $S_0(x_0) = y_0 \Leftrightarrow S_0(0) = 1 \Leftrightarrow \boxed{D_0 = 1}$

$S_1(x_1) = y_1 \Leftrightarrow S_1(1) = 2 \Leftrightarrow \boxed{D_1 = 2}$

$S_1(x_2) = y_2 \Leftrightarrow S_1(3) = 4 \Leftrightarrow 8A_1 + 4B_1 + 2C_1 + 2 = 4$

$\Leftrightarrow \boxed{4A_1 + 2B_1 + C_1 = 1} - *^1$

$\boxed{2}$ $S_0(x_1) = S_1(x_1) \Leftrightarrow S_0(1) = S_1(1)$

$\Leftrightarrow A_0 + B_0 + C_0 + 1 = 2$

$\Leftrightarrow \boxed{A_0 + B_0 + C_0 = 1} - *^2$

$\boxed{3}$ $S_0'(x_1) = S_1'(x_1) \Leftrightarrow S_0'(1) = S_1'(1) \Leftrightarrow \boxed{3A_0 + 2B_0 + C_0 = C_1} - *^3$

$\boxed{4}$ $S_0''(x_1) = S_1''(x_1) \Leftrightarrow S_0''(1) = S_1''(1) \Leftrightarrow 6A_0 + 2B_0 = 2B_1$

$\Leftrightarrow \boxed{3A_0 + B_0 = B_1} - *^4$

For natural cubic spline $\Rightarrow$

$S''(a) = S_0''(x_0) = S_0''(0) = 0 \quad \Leftrightarrow \quad \boxed{B_0 = 0}$

$S''(b) = S_1''(x_2) = S_1''(3) = 0 \quad \Leftrightarrow \quad 12A_1 + 2B_1 = 0$

$\Leftrightarrow \quad \boxed{B_1 = -6A_1} \; *^5$

- Solving $*^1, *^2, *^3, *^4, *^5$ gives $\boxed{A_0 = A_1 = B_1 = 0}$ and $\boxed{C_0 = C_1 = 1}$
- Hence, the natural cubic spline becomes linear:

$$S(x) = \begin{cases} S_0(x) = x + 1 & , 0 \le x \le 1 \\ S_1(x) = x + 1 & , 1 \le x \le 3 \end{cases}$$

$$= x + 1 \quad \text{on} \quad 0 \le x \le 3$$

Exp Consider the following function:

$$S(x) = \begin{cases} S_0(x) = x^3 + x - 1, & 0 \le x \le 1 \\ S_1(x) = 1 + C(x-1) + D(x-1)^2 - (x-1)^3, & 1 \le x \le 2 \end{cases}$$

[a] Find the constats C and D that makes $S(x)$ cubic spline.

[b] Is $S(x)$ natural cubic spline?

$$S'(x) = \begin{cases} S_0'(x) = 3x^2 + 1, & 0 \le x \le 1 \\ S_1'(x) = C + 2D(x-1) - 3(x-1)^2, & 1 \le x \le 2 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6x, & 0 \le x \le 1 \\ S_1''(x) = 2D - 6(x-1), & 1 \le x \le 2 \end{cases}$$

[a] • $S(x)$ is continuous at $x_1 = 1$ $\Leftrightarrow S_0(1) = S_1(1)$

$\Leftrightarrow 1 = 1$ does not help

• $S(x)$ is differentiable at $x_1 = 1$ $\Leftrightarrow S_0'(1) = S_1'(1)$

$\Leftrightarrow \boxed{4 = C}$

• $S(x)$ is twice diff. at $x_1 = 1$ $\Leftrightarrow S_0''(1) = S_1''(1)$

$\Leftrightarrow 6 = 2D \Leftrightarrow \boxed{D = 3}$

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[b] We check if $S_0''(x_0) = S_0''(0) \overset{?}{=} 0 \Rightarrow S_0''(0) = (6)(0) = 0$

and $S_1''(x_2) = S_1''(2) \overset{?}{=} 0 \Rightarrow S_1''(2) = 2(3) - 6(2-1) = 0$

so the cubic spline $S(x)$ is natural.