5.2 Series Solutions Near an Ordinary Point, Part I

In Chapter 3 we described methods of solving second order linear differential equations with constant coefficients. We now consider methods of solving second order linear equations when the coefficients are functions of the independent variable. In this chapter we will denote the independent variable by x. It is sufficient to consider the homogeneous equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \qquad (1)$$

since the procedure for the corresponding nonhomogeneous equation is similar.

Many problems in mathematical physics lead to equations of the form (1) having polynomial coefficients; examples include the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant.

Ordinary and Singular Points:

Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (*)$$

is put into standard form

$$y'' + P(x)y' + Q(x)y = 0$$

by dividing by the leading coefficient $a_2(x)$. We have the following definition.

DEFINITION: Ordinary and Singular Points

A point $x = x_0$ is said to be an **ordinary point** of the differential equation (*) if both $P(x)$ and $Q(x)$ are analytic at $x = x_0$. A point that is not an ordinary point is said to be a **singular point** of the equation.

Existence of power series solution about ordinary point:

THEOREM: Let $x = x_0$ be an **ordinary point** of the linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Then we can always find two linearly independent solutions in the form of a power series about the ordinary point $x = x_0$, that is, $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$.

A series solution converges at least on some interval $|x - x_0| < \rho$, whereas ρ is the distance from x_0 to the closest singular point.

EXAMPLE 1 Determine a lower bound for the radius of convergence of the series solution about $x = 0$ for the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad \alpha \text{ is a constant.}$$

SOLUTION:

Here, $x_0 = 0$ is an ordinary point, since $P(x) = -\frac{2x}{(1-x^2)}$ and $Q(x) = \frac{\alpha(\alpha+1)}{(1-x^2)}$ are analytic at $x_0 = 0$.

Also, $P(x)$ and $Q(x)$ have singular points at $x = \pm 1$.

Therefore, by the above Theorem, there exists power series solution about the ordinary point $x = 0$ of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$.

The radius of convergence of this series solution is the distance between the ordinary point $x = 0$ and the closest singular point. Hence, the radius of convergence is at least $\rho = 1$.

Thus, the Legendre equation has power series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ which converges for $|x| < 1$.

Method of finding series solution near an ordinary point of ODEs with polynomial coefficients:

Given the linear ODE with polynomial coefficients of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

- Choose the ordinary point to center the series. If 0 is available, it often works nicely.
- Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ and take necessary derivatives and substitute in the ODE.
- Shift indices to write the equation as one series in $(x - x_0)^n$.
- Compare the coefficients and get the recurrence relation using the **Identity Property** given in the review section.
- Use recurrence relation to find all c_n's in terms of c_0 and c_1. These give the two independent solutions.
- Write the general solution in the form of a series or write the first few terms.

EXAMPLE 2 Solve the following ODE.

$$y'' + xy = 0$$

SOLUTION:

- Choosing ordinary point x_0

  - We choose $x_0 = 0$.

- Considering series solution y and substituting y, y', y'' in the ODE

  - Take $y = \sum_{n=0}^{\infty} c_n x^n$.
  - Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.
  - Substituting in the ODE gives

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n = 0.$$

• Simplifying & shifting index

- The above equation can be written as

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

Put $k = n - 2$

Put $k = n + 1$

Writing each series in the form involving x^k

- So shifting the indices of each series gives

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k-1} x^k = 0.$$

• Writing as single series

- This implies

$$2c_2 + \sum_{k=1}^{\infty}[c_{k+2}(k+2)(k+1) + c_{k-1}]x^k = 0.$$

• Comparing coefficients of powers of x

- Comparing coefficients of powers of x gives

$$c_2 = 0,$$

$$c_{k+2}(k+2)(k+1) + c_{k-1} = 0 \quad \text{for } k \geq 1.$$

- Which gives

$$c_2 = 0,$$

$$c_{k+2} = -\frac{1}{(k+2)(k+1)} c_{k-1} \quad \text{for } k \geq 1.$$

Recurrence relation to determine coefficients c_n

- Finding coefficients c_n's

- $k=1 \Rightarrow \quad c_3 = -\frac{c_0}{2\cdot 3} = -\frac{c_0}{6}.$
- $k=2 \Rightarrow \quad c_4 = -\frac{c_1}{4\cdot 3} = -\frac{c_1}{12}.$
- $k=3 \Rightarrow \quad c_5 = -\frac{c_2}{5\cdot 4} = -\frac{0}{20} = 0.$
- $k=4 \Rightarrow \quad c_6 = -\frac{c_3}{6\cdot 5} = -\frac{1}{30}\left(-\frac{c_0}{6}\right) = \frac{c_0}{180}.$

$k=5 \Rightarrow \quad c_7 = -\frac{c_4}{7\cdot 6} = -\frac{1}{42}\left(-\frac{c_1}{12}\right) = \frac{c_1}{504}.$

$k=6 \Rightarrow \quad c_8 = -\frac{c_5}{8\cdot 7} = -\frac{1}{56}(-\frac{c_2}{20}) = 0.$

- Writing general solution

- $y = \sum_{n=0}^{\infty} c_n x^n$ implies

y	$=$	$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$
	$=$	$c_0 + c_1 x \ - \frac{c_0}{6}x^3 - \frac{c_1}{12}x^4 + \frac{c_0}{180}x^6 + \frac{c_1}{504}x^7 + \cdots$
	$=$	$c_0(1 - \frac{x^3}{6} + \frac{x^6}{180} - \cdots) + c_1(x - \frac{x^4}{12} + \frac{x^7}{504} - \cdots)$

- Writing solution in better form

May not be possible always

- We have found the solution $y = c_0 y_1(x) + c_1 y_2(x)$ where

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2\cdot 3)(5\cdot 6)\cdots((3n-1)3n)} x^{3n},$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(3\cdot 4)(6\cdot 7)\cdots((3n)(3n+1))} x^{3n+1}.$$

EXAMPLE 3 Solve the following ODE about the ordinary point $x_0 = 0$.

$$(x^2 + 1)y'' + xy' - y = 0$$

SOLUTION:

- Considering series solution y and substituting y, y', y'' in the ODE

  - Take $y = \sum_{n=0}^{\infty} c_n x^n$.
  - Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.
  - Substituting in the ODE gives

$$(x^2 + 1) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0.$$

- Simplifying & shifting index

- The above equation can be written as

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0.$$

Put $k = n$ | Put $k = n - 2$ | Put $k = n$ | Put $k = n$

Writing each series in the form involving x^k

- So shifting the indices of each series gives

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0.$$

- Writing as single series

- Above equation implies

$$2c_2 - c_0 + (6c_3 + c_1 - c_1)x + \sum_{k=2}^{\infty} [(k^2-1)c_k + (k+2)(k+1)c_{k+2}]\, x^k = 0.$$

• Comparing coefficients of powers of x

- Comparing coefficients of powers of x gives

$$2c_2 - c_0 = 0,$$

$$6c_3 = 0,$$

$$(k^2 - 1)c_k + (k + 2)(k + 1)c_{k+2} = 0 \quad \text{for } k \geq 2.$$

- Which gives

$$c_2 = \frac{c_0}{2},$$

$$c_3 = 0,$$

$$c_{k+2} = -\frac{k-1}{k+2}c_k \quad \text{for } k \geq 2.$$

Recurrence relation to determine coefficients c_n

• Finding coefficients c_n's

- $k = 2 \Rightarrow \quad c_4 = -\frac{1}{4}c_2 = -\frac{1}{4}\left(\frac{1}{2}c_0\right) = -\frac{1}{8}c_0.$
- $k = 3 \Rightarrow \quad c_5 = -\frac{2}{5}c_3 = 0.$
- $k = 4 \Rightarrow \quad c_6 = -\frac{3}{6}c_4 = -\frac{1}{2}(-\frac{1}{8}c_0) = \frac{1}{16}c_0.$
- $k = 5 \Rightarrow \quad c_7 = 0.$
- $\vdots$

• Writing general solution

- $y = \sum_{n=0}^{\infty} c_n x^n$ implies

y	$=$	$c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$
	$=$	$c_0 + c_1x + \frac{c_0}{2}x^2 - \frac{1}{8}c_0x^4 + \frac{1}{16}c_0x^6 + \cdots$
	$=$	$c_0(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots) + c_1x$

General solution