

Chapter 3: Solution of Linear Systems.

To solve a system we have 5 direct methods.

- 1) Gaussian Elimination : $[A|b] \rightarrow [U|c] + \text{Back substitution}$
- 2) $(A|b) \rightarrow (I|c)$: Gauss - Jordan elimination
- 3) Inverse method : $Ax = b \rightarrow x = A^{-1}b$.
- 4) $x_i = \frac{|A_i|}{|A|}$: Cramer's method.
- 5) $A = LU$. Factorization.

Speediness : $4 < 3 < 2 < 1 < 5$

To solve linear & Nonlinear system numerically, there exist 3 ways:

- 1) Fixed point iteration
- 2) Gauss Seidel method.
- 3) Newton method.

3.3 Upper triangular Linear system:

Back substitution:

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$4x_2 + 6x_3 = 10$$

$$10x_3 = 10$$

Using B.S, we found $x_3 = 1$, $x_2 = 1$, $x_1 = 1$

The Augmented matrix
$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 0 & 4 & 6 & 10 \\ 0 & 0 & 10 & 10 \end{array} \right]$$

Def: An $n \times n$ matrix $A = [a_{ij}]$ is called an upper triangular if $a_{ij} = 0$, $\forall i > j$, and is called Lower triangular if $a_{ij} = 0$, $\forall i < j$

Total Cost for solving 3×3 systems of upper triangular using B.S

$$\left[\begin{array}{ccc|c} - & - & - & - \\ 0 & - & - & - \\ 0 & 0 & - & - \end{array} \right]$$

Step	+, -	x, ÷
1	-	1
2	1	2
3	2	3

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In General: If we have upper triangular system of L.E.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n-1}x_{n-1} + a_{1n}x_n = b_1$$

$$a_{22}x_2 + \dots + a_{2n-1}x_{n-1} + a_{2n}x_n = b_2$$

⋮

$$a_{nn}x_n = b_n$$

Then the cost will be as follow:

Then the cost of solving $n \times n$ system, will be:

• Cost $(+, -) :=$

$$= \sum_{k=1}^n (k-1)$$

$$= \sum_{k=1}^n k - \sum_{k=1}^n 1$$

$$= \frac{n(n+1)}{2} - n$$

• Cost $(x, \div) = \sum_{k=1}^n k = \frac{n(n+1)}{2}$

same as F.S
cost = n^2

$$\Rightarrow \text{Total Cost for B.S} = \frac{n(n+1)}{2} + \frac{n(n+1)}{2} - n = n^2$$

(S2)

Step	+, -	x, ÷
1	—	1
2	1	2
3	2	3
⋮	⋮	⋮
k	k-1	k
⋮	⋮	⋮
n	n-1	n

3.4 Gaussian Elimination & pivoting:

$$Ax = b$$

$$[A|b] \rightarrow [U|c] + \text{B.S.}$$

We can use Row Operations:

- 1) Multiply a row by nonzero scalar
- 2) Switch any two rows.
- 3) Replace any row by adding to it a nonzero multiple by another row

Using the following equation:

$$\text{row } r_{\text{new}} = \text{row } r_{\text{old}} - m_{rp} \text{ row } p.$$

where $m_{rp} = \frac{a_{rp}}{a_{pp}}, \quad r > p.$

Example:

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$

$$2x_1 + 4x_3 + 3x_4 = 28$$

$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$

$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6$$

First we write the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array} \right]$$

Here the pivot row
is the 1st row.

we have $m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$

$m_{31} = \frac{a_{31}}{a_{11}} = \frac{4}{1} = 4$

$m_{41} = \frac{a_{41}}{a_{11}} = \frac{-3}{1} = -3$

} 3 divisions
⇒ in first step

→ $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array} \right]$

To get this matrix,
we multiplied the first
row by m_{rp} , then
adding it to row p .

So we multiplied 1st row (3 times) i.e. (4×3)

then add 1st row to (the other rows) (i.e.) (4×3)

then the cost for first step is

$(4 \times 3) \times 3$ for $\times, \div$ & (4×3) for $+, -$

Again, continue in Gaussian elimination, the final Augmented matrix is:

→ $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array} \right]$

Solving using B.S

we have:

$x_4 = 2, x_3 = 4$

$x_2 = -1, x_1 = 3$

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Then the total Cost for Reducing 4×4 into upper system

$\Rightarrow$ the cost for Reducing

is $20 + 26 = 46$

Then for Total for Solving
the L.S is

$$46 + (4)^2 = 62$$

(G.E + B.S)

step	+, -	$\times, \div$
1	4×3	$(4 \times 3) + 3$
2	3×2	$(3 \times 2) + 2$
3	2×1	$(2 \times 1) + 1$
Total	20	26 multipliers m.

In General: The Cost of Reducing $A_{n \times n}$ into upper triangular:

the Cost for (+, -)

$$= \sum_{k=1}^{n-1} (n-k)(n-k+1)$$

$$= \sum_{k=1}^{n-1} n^2 + n + k^2 - (2n+1)k$$

$$= \sum_{k=1}^{n-1} (n-k)(n-k) + (n-k)$$

$$= \sum_{k=1}^{n-1} (n-k)^2 + \sum_{k=1}^{n-1} (n-k)$$

$$= \frac{n^3 - n}{3}$$

step	+, -	$\times, \div$
1	$(n-1) \times n$	$(n-1) \times n + (n-1)$
2	$(n-2) \times (n-1)$	$(n-2) \times (n-1) + (n-2)$
3	$(n-3) \times (n-2)$	$(n-3) \times (n-2) + (n-3)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k)(n-k+1)$	$(n-k)(n-k+1) + (n-k)$
$\vdots$	$\vdots$	$\vdots$
n-1	1×2	$(1 \times 2) + 1$

Now we know that we can do change of variables (i.e)

Let $p = (n-k)$, therefore: $\sum_{p=1}^n p^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{p=1}^n p = \frac{n(n+1)}{2}$

⇒ Cost for (+, -) becomes:

$$\sum_{p=1}^{n-1} p^2 + \sum_{p=1}^{n-1} p, \text{ therefore:}$$

$$= \frac{(n-1)n(2(n-1)+1)}{6} + \frac{(n-1)((n-1)+1)}{2}$$

$$= \frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2} = \frac{n^3 - n}{3}$$

Total cost for (X, ÷)

$$= \sum_{k=1}^{n-1} (n-k) + (n-k)(n-k+1)$$

$$= \underbrace{\sum_{k=1}^{n-1} (n-k)}_{\frac{n^2}{2} - \frac{n}{2}} + \underbrace{\sum_{k=1}^{n-1} (n-k)(n-k+1)}_{\frac{n^3 - n}{3}} \xrightarrow{\text{take } n-k=p} \sum_{p=1}^{n-1} p^2 + \sum_{p=1}^{n-1} p$$

$$= \frac{(n-1)n}{2} + \frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2} = \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$$

∴ Total Cost for Reducing $n \times n$ system into upper triangular:

$$= \frac{(n-1)n}{2} + \frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2} + \frac{(n-1)n(n-1)}{6} + \frac{(n-1)n}{2}$$

$$= \text{Reducing } \frac{4n^3 + 3n^2 - 7n}{6} \approx \frac{2}{3} n^3$$

Finally total Cost for Solving system using Gaussian Elimination:

$$= \frac{4n^3 + 3n^2 - 7n}{6} + n^2 = \frac{2}{3} n^3 + \frac{3}{2} n^2 - \frac{7}{6} n \approx \frac{2}{3} n^3$$

Pivoting to Avoid $a_{pp}^{(p)} = 0$

Pivoting means to find the element in the row that is non zero element to do some calculation (Gaussian-Jordan)

Assume $A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix}$

If $a_{pp}^{(p)} = 0$, then row p can't be used to eliminate the elements in Column p below the main diagonal therefore we find row k , where $a_{kp}^{(p)} \neq 0$ and we change rows, this criteria is called Pivoting strategy.

- Trivial pivoting: If $a_{pp}^{(p)} \neq 0$, we don't change rows otherwise we have to change with row with $a_{kp}^{(p)} \neq 0$.

Pivoting is used to Reduce the Error.

Because the computer uses fixed-precision arithmetic, it is possible that a small error will be introduced each time that an arithmetic operation is performed

Example: The values $x_1 = x_2 = 1.000$ are the solutions of

$$1.133 x_1 + 5.281 x_2 = 6.414$$

$$24.14 x_1 - 1.210 x_2 = 22.93$$

Use four digit arithmetic and Gaussian elimination with trivial pivoting to find an approximation for the solution.

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array} \right]$$

$$m_{21} = \frac{24.14}{1.133} \approx 21.31, \text{ where } m_{rp} = \frac{a_{rp}}{a_{pp}}, r > p$$

$$\& \text{ row 2} = \text{row 2} - m_{21} \cdot \text{row 1}$$

$$\Rightarrow a_{22}^{(2)} = -1.210 - 21.31(5.281) = -113.7$$

$$a_{23}^{(2)} = 22.93 - 21.31(6.414) = -113.8$$

$$\Rightarrow \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array} \right] \Rightarrow \text{B.S to find } x_1 \& x_2$$

$$\Rightarrow x_1 = 0.9956 \& x_2 = 1.001.$$

(The error starts to appear from m_{21}).

We will solve the same example using changing the rows:

$$24.14 x_1 - 1.210 x_2 = 22.93$$

$$1.133 x_1 + 5.281 x_2 = 6.414$$

$$m_{21} = \frac{1.133}{24.14} \approx 0.04693$$

$$\Rightarrow a_{22}^{(2)} = 5.281 - (0.04693)(-1.210) \approx 5.338$$

$$a_{23}^{(2)} = 6.414 - (0.04693)(22.93) \approx 5.338$$

$$\Rightarrow \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array} \right] \Rightarrow \text{P.S. to solve } x_1, x_2$$

$$\Rightarrow x_2 = x_1 = 1$$

Note: the error in the first example is due to the magnitude of the multiplier $m_{21} \approx 21.31$

- Example (1) was without partial pivoting
- Example (2) was with partial pivoting.

Therefore we will use two ways to avoid propagation of error.

1) Partial Pivoting: we take the pivot element to be the largest (In magnitude) in the Remaining Column P (on the diagonal and below), then we locate row K

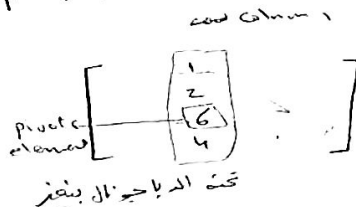
that is :

$$|a_{kp}| = \max \{ |a_{pp}|, |a_{(p+1)p}|, \dots, |a_{(n-1)p}|, |a_{np}| \}$$

then we change row p with row k , if $k > p$

In this case the multipliers $m_{rp} < 1$.

(see the last example).



2) Scaled Partial Pivoting: (To avoid further error propagation)

A scalar factor must be computed first for each row (equation). Define: $S_i = \max_{1 \leq j \leq n} |a_{ij}|$, ($1 \leq i \leq n$)

then we use the row for which

the ratio $\frac{|a_{i,p}|}{S_i}$ is greatest.

as the pivotal row.

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix}$$

أبسط (الأكبر) من كل s_i

Example:

$$0.0001x + y = 1$$

$\Rightarrow$

$$s_1 = 1$$

$$s_2 = 1$$

$$x + y = 2$$

أبسط (الأكبر) من كل s_i

Now we take the ratio: $\frac{0.0001}{1} = 0.0001 < \frac{1}{1}$

so the second equation is pivoting equation.

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(2) Gauss-Jordan Elimination:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array} \right]$$

Example: Solve:

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ x_1 - 2x_2 + 3x_3 &= 2 \\ 3x_1 + 4x_2 - x_3 &= 6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array} \right] \xrightarrow{\div 3} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{4}{3} & 3 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{4}{3} & 3 \\ 0 & -\frac{8}{3} & \frac{5}{3} & -1 \\ 0 & 2 & -5 & -3 \end{array} \right] \rightarrow \dots$$

then the Cost for ^{Solving} Reducing 3×3 matrix using Gauss-Jordan Reduction is equal: 30

steps	+, -	$\times, \div$
1	3×2	$3 \times 2 + 3$
2	2×2	$2 \times 2 + 2$
3	1×2	$1 \times 2 + 1$

In General: The Total Cost for ^{solving} Reducing $n \times n$ matrices is equal.

Steps	+, -	x, ÷
1	$n \times (n-1)$	$n \times (n-1) + n$
2	$(n-1) \times (n-1)$	$(n-1) \times (n-1) + (n-1)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k+1)(n-1)$	$(n-k+1)(n-1) + (n-k+1)$
$\vdots$	$\vdots$	$\vdots$
n	$1 \times (n-1)$	$(n-1) \times 1 + 1$

$$\sum_{k=1}^n 2((n-k+1)(n-1)) + (n-k+1)$$

$$= (1 + 2(n-1)) \left(\sum_{k=1}^n (n-k+1) \right)$$

$$= (2n-1) \sum_{k=1}^n (n-k+1) = (2n-1) \left(n^2 - \frac{n(n+1)}{2} + n \right)$$

$$= \frac{2n^3 + n^2 - n}{2}$$

Note that if we compare Gaussian elimination with Gauss-Jordan Reduction for large n

$$\text{(Gauss-Jord)} \quad \frac{2n^3 + n^2 - n}{2} > \frac{4n^3 + 3n^2 - 7n}{6} + n^2 \text{ (Gaussian)}$$

$$\Rightarrow n^3 > \frac{2}{3}n^3 \quad n \rightarrow \infty$$

$$= \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

this implies that Gaussian is faster than Gauss Jordan Reduction. Cost for solving $\downarrow$:

$$\text{Gauss} \quad \frac{2n^3 + n^2 - n}{2} > \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n \quad \text{Gaussian}$$

(3) Inverse method:

$$A x = b \Rightarrow A^{-1} b = x$$

$$[A | I] \rightarrow [I | A^{-1}]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{bmatrix} A^{-1}$$

for 3x3 matrix, the cost is

equal

$$\sum (+, -) + \sum (x, \div) + \text{Cost}(A^{-1}b)$$

$$\text{Cost of } (A^{-1}b) = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Rightarrow \text{Cost}(A^{-1}b) = 3 (3 \text{ multiplication} + 2 \text{ add})$$

$$\Rightarrow \text{Total Cost} = 60 + 15 = 75$$

In General: Cost for nxn system:

$$\Rightarrow \text{Total Cost} = 3n^3 - \frac{5}{2}n^2 + \frac{1}{2}n$$

(+, -, x, ÷)

$$\& \text{Cost}(A^{-1}b) = n(n \text{ multip.} + (n-1) \text{ add})$$

$$= n(2n-1)$$

$$\Rightarrow \text{Total Cost} = 3n^3 - \frac{7n^2}{2} + \frac{1}{2}n$$

check !!

$$= 3n^3 - \frac{n^2}{2} - \frac{1}{2}n$$

Steps	+, -	x, ÷
1	5x2	5x2+5
2	4x2	4x2+4
3	3x2	3x2+3

Steps	+, -	x, ÷
1	(2n-1)x(n-1)	(2n-1)x(n-1)+(2n-1)
2	(2n-2)x(n-1)	(2n-2)(n-1)+(2n-2)
...	...	...
k	(2n-k)(n-1)	(2n-k)(n-1)+(2n-k)
...	...	...
n	n(n-1)	n(n-1)+n
	$\frac{3}{2}n^3 - 2n^2 + \frac{1}{2}n$	$\frac{3}{2}n^3 - \frac{n^2}{2}$

(4) Cramer's Rule:

$$|x_i| = \frac{|A_i|}{|A|}, \quad |A| \neq 0$$

Recall: for 2×2 , $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

Note: Cramer's is faster than the Inverse in 2×2 matrix.

- we used it if we are interested in one of the x_i 's
 $\frac{|A_1|}{|A|}, \frac{|A_2|}{|A|}$
- Cost of ^{solving} 2×2 system matrix = $3(2 \text{ multipl} + 1 \text{ add}) + 2 \text{ division}$
 $= 11$

- Cost for ^{solving} 3×3 system matrix = $4 \text{ determinants} + 3 \text{ division}$
 $\frac{|A_1|}{|A|}, \frac{|A_2|}{|A|}, \frac{|A_3|}{|A|}$

$$\begin{vmatrix} 3 & -5 & -2 \\ \vdots & \vdots & \vdots \end{vmatrix} = 4 \left(3(D_2) + 3 \text{ multipl.} + 2 \text{ add} \right) + 3$$
$$= 4(3(3) + 3 + 2) + 3 = 59.$$

Note: Cramer's is the slowest (the most cost) comparing to the other methods.

^{Total} Cost for ^{solving} $n \times n$ system: $(n+1) |n \times n| + n$

$$= (n+1) \left[n! \sum_{k=1}^{n-1} \frac{1}{k!} + (n! - 1) \right] + n$$

In General: by Induction: (The cost for 1 determinant)

$$\text{the cost} := n! \sum_{k=1}^{n-1} \frac{1}{k!}, (x, \div)$$

$$\& (n! - 1), (+, -).$$

proof: by Induction: when $n=2$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

requires: 2 multip. + 1 addition

$$\Rightarrow 2! \sum_{k=1}^1 \frac{1}{k!} = 2 \quad (x, \div) \quad \& \quad 2! - 1 = 1$$

then it's true for $n=2$. assume that it's true for m .

Let A be $(m+1) \times (m+1)$, then

$$\det(A) = \sum_{j=1}^{m+1} a_{ij} A_{ij}, \quad \forall \quad 1 \leq i \leq m+1$$

to compute A_{ij} , we need $\left(m! \sum_{k=1}^{m-1} \frac{1}{k!}\right) + (m! - 1)$

$$\Rightarrow \# \text{ of multiplication for } |A| \text{ is } \overset{\# \text{ of } A_{ij}}{(m+1)} \left[m! \sum_{k=1}^{m-1} \frac{1}{k!} \right] + \overset{a_{ij} \cdot A_{ij}}{(m+1)}$$

$$= (m+1)! \left[\sum_{k=1}^{m-1} \frac{1}{k!} + \frac{1}{m!} \right] = (m+1)! \sum_{k=1}^m \frac{1}{k!}$$

$$\& \text{ for addition: } \overset{\# \text{ of } A_{ij}}{(m+1)} \overset{2^{1, \dots}}{[m! - 1]} + \underline{m} = (m+1)! - 1.$$

□

3.5

(5) L U Factorization

" Only specific matrices can be solved using this method "

$$\begin{bmatrix} 4 & 3 & -1 & : & 21 \\ 2 & 0.5 & 2 & : & 4 \\ 1 & 0.25 & 3 & : & 0 \end{bmatrix} \rightarrow \text{we write } A = LU$$

$$U = \begin{bmatrix} 4 & 3 & -1 \\ 0 & -1 & 2.5 \\ 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix}$$

فكر في المصفوفة
بأنها مصفوفة

We solve: $Ax = b \Rightarrow LUx = b.$

Let $Ux = y$, then solve $Ly = b$ (Forward Substitution)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 21 \\ 4 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= 21 \\ y_2 &= -6.5 \\ y_3 &= -2 \end{aligned}$$

Now solve $Ux = y$ (Backward Substitution).

$$\begin{bmatrix} 4 & 3 & -1 \\ 0 & -1 & 2.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ -6.5 \\ -2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= 2 \\ x_2 &= 4 \\ x_3 &= -1 \end{aligned}$$

Note: This method is better than Gaussian elimination,

Since if b is changed, the change occurs in step [2] not from the beginning.

$LU > \text{Gaussian} > \text{Gauss-Jordan} > \text{Inverse} > \text{Cramer's}$

Cost for factoring $A_{n \times n}$ into LU factorization.

First: the Cost for Reducing 3×3 system to LU factorization

$$\Rightarrow \text{Cost } (A \rightarrow U) = 13$$

then Cost for solving the system

$$= U + L + B.S + F.S.$$

$$= 13 + 0 + 9 + 6 = 28.$$

$$n^2 - n \text{ steps } L \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \sim 8$$

Steps	+, -	x, ÷
1	2×2	$2 \times 2 + 2$
2	1×1	$1 \times 1 + 1$

(b up to X)

Note: If we use more than (1) b, then using LU factorization

the cost for $b_1 = 28$, but for $b_2, b_3, \dots = 15$

Since the change starts from step 2, that's why

it's better to use LU factorization

In General: Cost for factoring $A_{n \times n}$ into LU

$$\Rightarrow \text{Cost: } \frac{2(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2}$$

$$= \frac{4n^3 - 3n^2 - n}{6}$$

$\Rightarrow$ Total Cost for solving system

$$= \frac{4n^3 - 3n^2 - n}{6} + 0 + (n^2 - n) + n^2$$

$$= \frac{4n^3 + 9n^2 - 7n}{6}$$

Cost of Gaussian elimination

steps	+, -	x, ÷
1	$(n-1) \times (n-1)$	$(n-1) \times (n-1) + (n-1)$
2	$(n-2) \times (n-2)$	$(n-2) \times (n-2) + (n-2)$
⋮	⋮	⋮
k	$(n-k) \times (n-k)$	$(n-k) \times (n-k) + (n-k)$
⋮	⋮	⋮
n-1	1×1	$1 \times 1 + 1$

(67)

• The cost for solving using Gaussian elimination

$$= \frac{4n^3 + 3n^2 - 7n}{6} + n^2 \stackrel{\text{B.W.S}}{=} \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

• The cost for solving other b using LU

$$= n^2 - n + n^2 = 2n^2 - n$$

$$= \text{BS} + \text{F.S}$$

• Cost $A^2 = ??$ $A_{3 \times 3}$

$$\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} \times \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$

$$3(3+3+3) + 3(2+2+2)$$