Convert SSR to transfer function/matrix

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$$\left.\begin{aligned}\dot{x}(t) &= A\,x(t) + B\,u(t)\\ y(t) &= C\,x(t) + D\,u(t)\end{aligned}\right\}\ \text{SSR "time-domain"}$$

Taking laplace transform (with zero I.C.)

$$\left.\begin{aligned} s\,X(s) - \cancel{x(0)}^{\,IC} &= A\,X(s) + B\,U(s) \quad \text{(i)}\\ Y(s) &= C\,X(s) + D\,U(s) \quad \text{(ii)}\end{aligned}\right\} \Rightarrow [\underbrace{s}_{\text{Variable}}\underbrace{I - A}_{\text{matrix}}]$$

$$\boxed{(sI - A)}\,X(s) = B\,U(s)$$

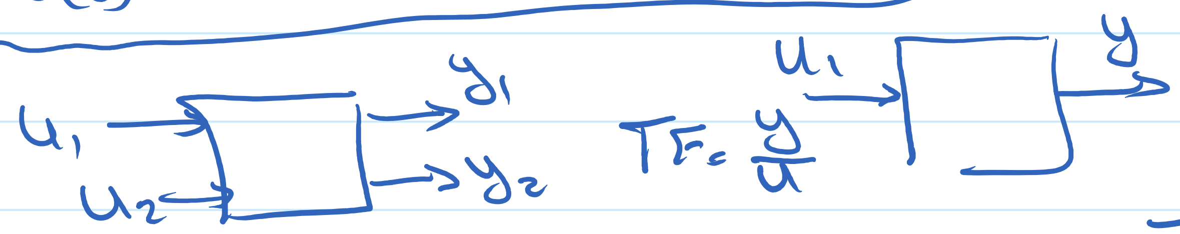
$$X(s) = (sI-A)^{-1} B\,U(s) \quad \text{— (iii)}$$

Sub Eq (iii) into Eq (ii)

$$Y(s) = C\left[(sI-A)^{-1} B\,U(s)\right] + D\,U(s)$$

$$Y(s) = \left[C(sI-A)^{-1}B + D\right]U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D} \quad \text{— (*)}$$



$$TM = \begin{bmatrix} \frac{y_1}{u_1} & \frac{y_1}{u_2} \\ \frac{y_2}{u_1} & \frac{y_2}{u_2} \end{bmatrix} = \underbrace{\begin{bmatrix} G_1(s) & G_2(s) \\ G_3(s) & G_4(s) \end{bmatrix}}_{\text{Transfer matrix}}$$

$n=2$
$r=1$
$P=1$

$n \times r$

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\# of input

Ex: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$

$P \times n$

$$\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

$$(sI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{s^2+3s+2} \quad \Delta$$

$$C(sI-A)^{-1} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}_{2\times 2} \frac{1}{\Delta} = \begin{bmatrix} s+3 & 1 \end{bmatrix} \frac{1}{\Delta}$$

$$C(sI-A)^{-1}B = \frac{1}{\Delta} \underbrace{\begin{bmatrix} s+3 & 1 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{2\times 1} = \frac{1}{\Delta}$$

$$C(sI-A)^{-1}B + D = \frac{1}{s^2+3s+2}$$

matlab

A = ✓ B = ✓ C = ✓ D = ✓

[num, den] = ss2tf(A, B, C, D) ← TF

G = tf(num, den) • First method

A = ✓ B ✓ C ✓ D ✓ Second method

syms s

{G = C * inv(s * eye(n) - A) * B + D}

Symbolic fun

– Note: The process of converting transfer fun to SSR is **NOT** unique. There are various "realizations" possible.

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$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$$

Classification of TF.

① strictly proper TF :- if deg $b(s) <$ deg $a(s)$
or equivalent $G(\infty) = 0$ $\quad G(s) = \frac{s+1}{s^2+3s+2}$

② improper T.F :- if deg $b(s) >$ deg $a(s)$

$$G(s) = \frac{s^3+3s+2}{s^2+10s} \leftarrow \times$$

③ Bi-proper TF :- if deg $b(s) =$ deg $a(s)$

$$G(s) = \frac{s^2+10s+2}{s^2+3s+2} \qquad G(\infty) \neq 0 =$$
$$G(\infty) = (\text{Constant})$$

* proper TF :- if $G(s)$ is said to be proper if deg $b(s) \leq$ deg $a(s)$

Transfer matrix →

$$G(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1r}(s) \\ \vdots & \ddots & \vdots \\ g_{p1}(s) & \cdots & g_{pr}(s) \end{bmatrix}$$

- if $g_{ij}(s)$ are strictly proper TF, therefore $G(s)$ is strictly proper Transfer matrix
- if some $g_{ij}(s)$ are strictly proper or bi-proper TF. Therefore $G(s)$ is proper T. M "transfer matrix"

Converting from TF or TM to SSR:-

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There are three Cases:-

① simple numerator (strictly Proper)

$$G(s) = \frac{y(s)}{u(s)} = \frac{1}{a_0 s^n + a_1 s^{n-1} + \dots + a_n}$$

Ex: Convert $G(s) = \dfrac{1}{s^3 + 10s^2 + 3s + 2}$ to SSR

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 10s^2 + 3s + 2}$$

$\rightarrow Y(s)(s^3 + 10s^2 + 3s + 2) = U(s)$ take the inverse of Laplace

$\rightarrow \dddot{y}(t) + 10\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$

Let the states are

$y(t) = x_1 \Rightarrow \dot{y}(t) = \dot{x}_1$
$\dot{y}(t) = x_2 \Rightarrow \ddot{y}(t) = \dot{x}_2$
$\ddot{y}(t) = x_3 \Rightarrow \dddot{y}(t) = \dot{x}_3$

$\boxed{\dot{x}_1 = x_2}$ — ⓐ

$\boxed{\dot{x}_2 = x_3}$ — ⓑ

$\boxed{\dot{x}_3 = u(t) - 10x_3 - 3x_2 - 2x_1}$ — ⓒ

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -10 \end{bmatrix}}_{A} X + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

From the states assumption

$y(t) = x_1$ ← output

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C\text{-vector row}} x(t) + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u(t)$$

$\dot{x} = Ax + Bu$

$y = Cx + Du$

Case 2:- Numerator less than denominator

Ex) = Convert $G(s) = \frac{s^2+2s+1}{s^3+3s^2+10s+2}$ (strictly proper)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^2+2s+1}{s^3+3s^2+10s+2} = \frac{N(s)}{D(s)}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{V(s)}{U(s)} \cdot \frac{Y(s)}{V(s)}$$

$$\frac{V(s)}{U(s)} = \frac{1}{D(s)} \quad ; \quad \frac{Y(s)}{V(s)} = N(s)$$

Case 1

$$\frac{V(s)}{U(s)} = \frac{1}{D(s)} = \frac{1}{s^3+3s^2+10s+2}$$

$V(s)(s^3+3s^2+10s+2) = U(s)$ — (*) take Laplace inv.

$$\dddot{v}(t) + 3\ddot{v}(t) + 10\dot{v}(t) + 2v(t) = u(t)$$

let the states are

$x_1 = v(t)$; $\boxed{\dot{x}_1 = x_2}$ — (a)

$x_2 = \dot{v}(t)$; $\boxed{\dot{x}_2 = x_3}$ — (b)

$x_3 = \ddot{v}(t)$

Sub the states in Eq (*)

$$\boxed{\dot{x}_3 = u(t) - 3x_3 - 10x_2 - 2x_1} \text{ — (c)}$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -10 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

$$\frac{Y(s)}{V(s)} = N(s) = s^2 + 2s + 1$$

$$Y(s) = V(s)\left(s^2 + 2s + 1\right)$$

take inverse Laplace

$$y(t) = \ddot{v}(t) + 2\dot{v}(t) + v(t) \quad (**)$$

Sub the states in Eq (**) Remember

$$x_1 = v(t)$$
$$x_2 = \dot{v}(t)$$
$$x_3 = \ddot{v}(t)$$

$$y(t) = x_3 + 2x_2 + x_1 \quad (2)$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u(s)$$

$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \to C$, $\begin{bmatrix} 0 \end{bmatrix} \to D$

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Case 3 :- biproper fun. $G(s) = \dfrac{10s^3+2s^2+10s+2}{s^3+3s^2+2s+5}$

$$s^3+3s^2+2s+5 \overline{\big)\, 10s^3+2s^2+10s+2}\quad 10$$

$$\underline{-\,\cancel{10s^3} \mp 30s^2 \mp 20s \mp 50}$$

$$-28s^2-10s-48$$

$G(s) = 10 + \dfrac{-28s^2-10s-48}{s^3+3s^2+2s+5}$

matrix D

state space Rep

Case 2 strictly proper fun

A, B, C matrices