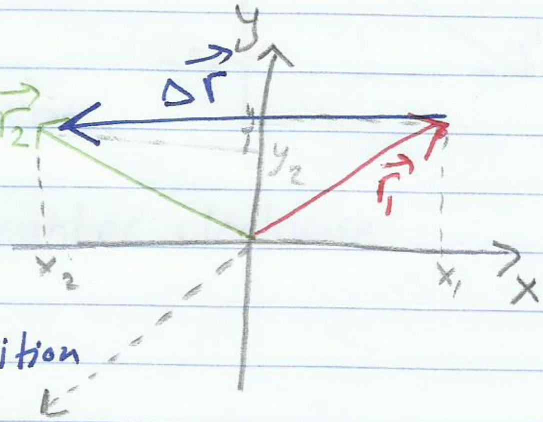
Chapter 4: Motion in Two and Three Dimensions

Position and Displacement:



$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Displacement = Change in Position

$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$

example: let $\vec{r}_1 = -3\hat{i} + 2\hat{j} + 5\hat{k}$ m
after 5s the particle moves from $\vec{r}_1$ to
$\vec{r}_2 = 5\hat{i} - 7\hat{j} + 13\hat{k}$ m
find $\Delta\vec{r}$? Displacement

$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$

$= (5\hat{i} - 7\hat{j} + 13\hat{k}) - (-3\hat{i} + 2\hat{j} + 5\hat{k})$

$\Delta\vec{r} = 8\hat{i} - 9\hat{j} + 8\hat{k}$ m

Sample Problem 4.01:

4.01 + 4.02 + 4.03 are related

A rabbit position is

$x = -0.31t^2 + 7.2t + 28$ m

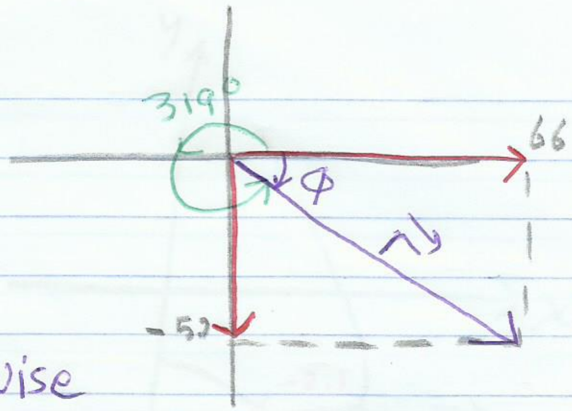
$y = 0.22t^2 - 9.1t + 30$ m

a) At $t = 15s$, What is the rabbit's Position vector $\vec{r}$?

$x = -0.31(15)^2 + 7.2(15) + 28 = -69.75 + 108 + 28$
$= 66.25$ m

$y = 0.22(15)^2 - 9.1(15) + 30 = 49.5 - 136.5 + 30$
$= -57$ m

$\vec{r} = 66\hat{i} - 57\hat{j}$



$\vec{r} = \sqrt{(66)^2 + (57)^2} = 87\ m$

$\tan\phi = \frac{y}{x}$

$\phi = \tan^{-1}\left(\frac{-57}{66}\right) = -41^\circ$

$\vec{r} = 87\ m$, at 41° with $+x$ clockwise

or

$\vec{r} = 87\ m$, at 319° with $+x$ Counter clockwise

Average Velocity:

$$\vec{V}_{avg} = \frac{\Delta\vec{r}}{\Delta t}\ m/s = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}$$

Instantaneous Velocity

$$\vec{V}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$\vec{V}_{inst} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Sample Problem 4.02:-

find the rabbit velocity at $t = 15\ s$.

$x = -0.31t^2 + 7.2t + 28\ m$

$y = 0.22t^2 - 9.1t + 30\ m$

$V_x = \frac{dx}{dt} = -0.31(2t) + 7.2$, $V_x = -0.62t + 7.2\ m/s$

$V_x(15) = -0.62(15) + 7.2$

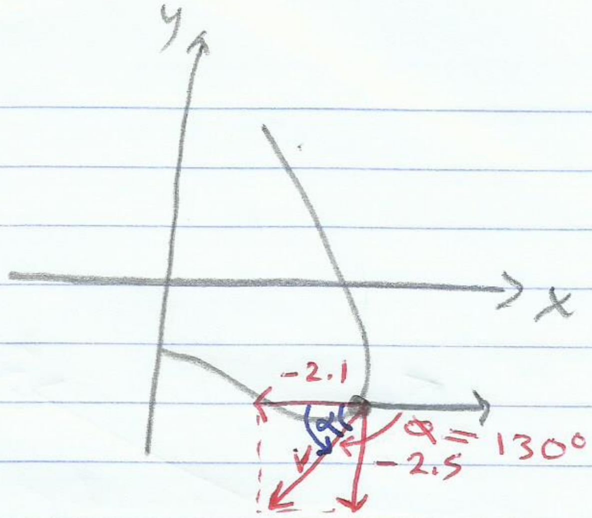
$= -2.1\ m/s$

$V_y = \frac{dy}{dx} = 0.22(2t) - 9.1 = 0.44t - 9.1$

$V_y(15) = 0.44(15) - 9.1$

$V_y = -2.5\ m/s$ at $t = 15s$

$\vec{V}(15) = -2.1\hat{i} - 2.5\hat{j}\ m/s$



$\vec{V}(15) = -2.1\hat{i} - 2.5\hat{j}$ m/s

$V(15) = \sqrt{(2.1)^2 + (2.5)^2}$

$= 3.26$ m/s

$\alpha = \tan^{-1}\left(\frac{2.5}{2.1}\right) = 50°$

$\vec{V}(15) = 3.26$ m/s, at $50°$ with $-x$ counter clockwise

or

$\vec{V}(15) = 3.26$ m/s, at $130°$ with $+x$ clockwise.

Problem: for the rabbit

$x = -0.31t^2 + 7.2t + 28$ m

$y = 0.22t^2 - 9.1t + 30$ m

Find the average velocity from $t_1 = 0$ to $t_2 = 15$ s

$x_0 = 28$ m $\longrightarrow x_{15} = 66$ m

$y_0 = 30$ m $\longrightarrow y_{15} = -57$ m

$\vec{r}_0 = 28\hat{i} + 30\hat{j} \longrightarrow \vec{r}_{15} = 66\hat{i} + {}^{-}57\hat{j}$ m

$\vec{V}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_{15} - \vec{r}_0}{15} = \frac{(66\hat{i} - 57\hat{j}) - (28\hat{i} + 30\hat{j})}{15}$

$= \frac{38\hat{i} - 87\hat{j}}{15}$

$\vec{V}_{avg} = 2.5\hat{i} - 5.8\hat{j}$ m/s

Average Acceleration:

$$\vec{a}_{avg} = \frac{\Delta \vec{V}}{\Delta t} \ m/s^2$$

Instantaneous Acceleration

$$\vec{a}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta \vec{V}}{\Delta t}$$

$$\vec{a}_{inst} = \frac{d\vec{V}}{dt} \ m/s^2$$

$$\vec{a}_{inst} = \frac{dV_x}{dt}\hat{i} + \frac{dV_y}{dt}\hat{j} + \frac{dV_z}{dt}\hat{k}$$

$$a_x = \frac{dV_x}{dt} \ ,, \ a_y = \frac{dV_y}{dt} \ ,, \ a_z = \frac{dV_z}{dt}$$

Sample Problem 4.03:

For the rabbit find the acceleration at $t = 15$ s.

$$x = -0.31t^2 + 7.2t + 28 \ m$$

$$V_x = \frac{dx}{dt}$$

$$V_x = -0.62t + 7.2 \ m/s$$

$$y = 0.22t^2 - 9.1t + 30 \ m$$

$$V_y = \frac{dy}{dt}$$

$$V_y = 0.44t - 9.1 \ m/s$$

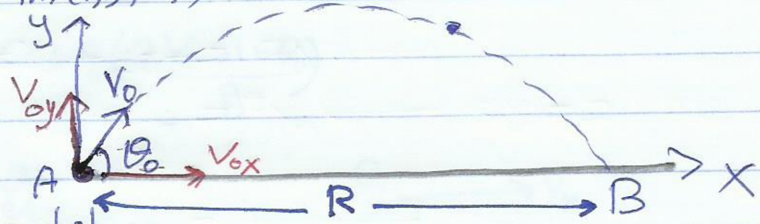
$$a_x = \frac{dV_x}{dt} = -0.62 \ m/s^2 \ \text{constant}$$

$$a_y = \frac{dV_y}{dt} = +0.44 \ m/s^2 \ \text{constant}$$

$$\vec{a} = -0.62\hat{i} + 0.44\hat{j} \ m/s^2$$

Projectile Motion:

It is a motion in 2 Dimensions



The initial velocity $\vec{V_o} = V_o$, at θ above the horizontal

or

$\vec{V_o} = V_{ox}\hat{i} + V_{oy}\hat{j}$; $V_{ox} = V_o \cos\theta$

$V_{oy} = V_o \sin\theta$

$\vec{a} = -9.8\hat{j}\ m/s^2 \Rightarrow a_x = 0$

$a_y = -9.8 m/s^2$ constant

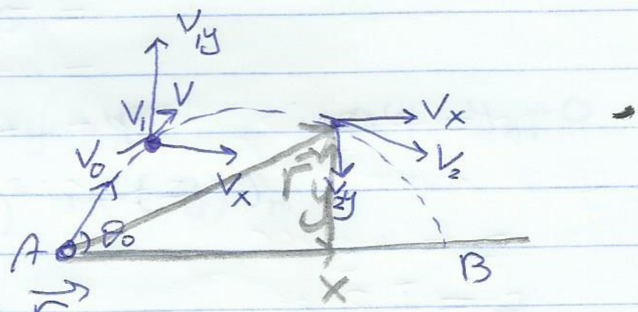
At any time (t) find:

1) $\vec{V}$? $V_x = V_{ox} = V_o \cos\theta$ ① , $a_x = 0$

V_x alway constant, there is No a_x

$V_y = V_{oy} + a_y t$

$V_y = V_o \sin\theta_o - gt$ ②



2) Find the position?

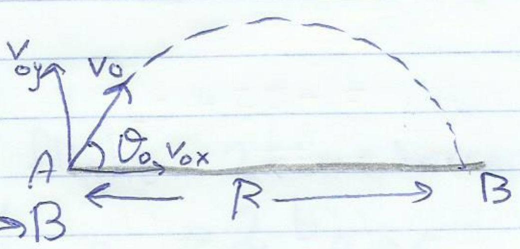
At any time the position $\vec{r}$ has 2 Components x + y.

$x = V_{ox} t = V_o \cos\theta_o t$ ③ V_x always constant.

$y = V_{oy} t + \frac{1}{2} a_y t^2$

$= V_o \sin\theta_o t + \frac{1}{2}(-g) t^2$ ④

Find the time of flight?



time of flight is the time for projectile to reach from $A \to B$

from $A \to B$ $\Delta y = 0$

$\Delta y = V_{oy} t + \frac{1}{2} a_y t^2$

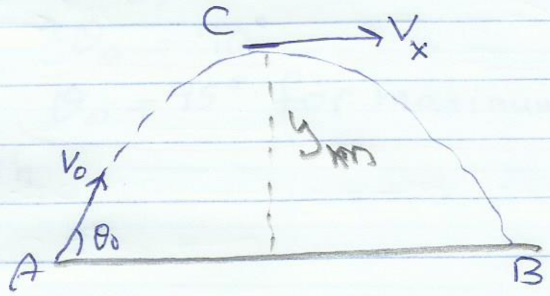
$0 = V_o \sin\theta_o t_f + \frac{1}{2}(-g) t_f^2 \Rightarrow t_f = \frac{2 V_o \sin\theta_o}{g}$

Find the Range?

Range = maximum X from A $\rightarrow$ B

$$\text{Range} = V_x \cdot t_f = V_0 \cos\theta_0 \left(\frac{2V_0 \sin\theta_0}{g}\right)$$

$$= \frac{V_0^2}{g}(2\sin\theta_0 \cos\theta_0)$$



$$\text{Range} = \frac{V_0^2}{g}\sin(2\theta_0)$$

Find $\vec{V}$ at maximum height?

At maximum height $= y_m$

$V_y = 0$

$V_x = V_0 \cos\theta_0$

Find the maximum height? y_m?

The motion from A $\rightarrow$ C

$(V_{0y})_A = V_0 \sin\theta_0$, $(V_y)_C = 0$

$a_y = -g$

$V_y^2 = V_{0y}^2 + 2a_y \Delta y$, $\Delta y = y_m - 0$

$0 = (V_0 \sin\theta_0)^2 + 2(-g)y_m$

$$y_m = \frac{V_0^2 \sin^2\theta_0}{2g}$$

Find the time to reach y_m?

$V_y = V_{0y} + a_y t$ (A $\rightarrow$ C)

$0 = V_0 \sin\theta_0 + (-g)t$

$$t = \frac{V_0 \sin\theta_0}{g}$$

Note; that the time of flight = 2 time to reach y_m

$t_{A\rightarrow B} = 2\, t_{A\rightarrow C}$

Problem: Find θ_0 for the Projectile to achieve maximum Rang?

$$Range = \frac{V_0^2 \sin(2\theta_0)}{g}$$

for maximum Range , $\sin(2\theta_0) = 1$

$$2\theta_0 = 90^\circ$$

$$\theta_0 = 45^\circ \text{ for maximum } R$$

Problem:- Find the angle at which $Range = y_{max}$

$$\frac{2V_0^2 \sin\theta_0 \cos\theta_0}{\cancel{g}} = \frac{V_0^2 \sin^2\theta_0}{\cancel{g}}$$

$$2 = \frac{\sin\theta_0}{\cos\theta_0}$$

$$\tan\theta_0 = 2$$

$$\theta_0 = 63^\circ$$

Now we will solve Problems 23, 33, 43.

Solve sample Problem 4.04

" " " 4.05

(4-33) V_0 at 53° with vertical

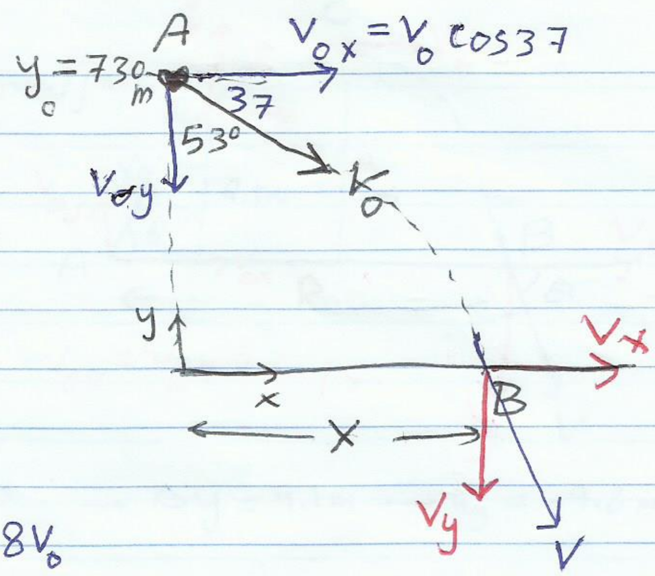
$y_0 = 730m$

$t = 5s$ time of flight

find: a) V_0?

b) X?

c) At the ground find V_x? V_y?



a) $V_{0x} \overset{+}{=} V_0 \cos 37 = {}^{+}0.8V_0$

$V_{0y} = {}^{-}V_0 \sin 37 = {}^{-}0.6V_0$

from y-motion $A \rightarrow B$

$V_{0y} = -0.6V_0$, $a_y = -9.8 m/s^2$, $\Delta y = -730m$

$$\Delta y = V_{0y}t + \frac{1}{2}a_y t^2$$

$$-730 = (-0.6V_0)(5) + \frac{1}{2}(-9.8)(5)^2$$

$$-730 = -3V_0 - 122.5$$

$$V_0 = \frac{730 - 122.5}{3} = \boxed{202.5 m/s}$$

$V_{0y} = -0.6(202.5) = -121.5 m/s$

b) X? x-motion from $A \rightarrow B$, $V_{0x} = 0.8(202.5) = 162 m/s$

$\Delta X = (V_{0x})t = (0.8V_0)(5) = (0.8)(202.5)(5)$

$X - X_0 = 810m$, $X_0 = 0$

$X = 810m$

c) At point B (the ground)

$V_x = V_{0x} = 162 m/s$

$V_y = V_{0y} + a_y t$

$= (-0.6V_0) + (-9.8)(5)$

$= (-121.5) - 49$

$V_y = -170.5 m/s$

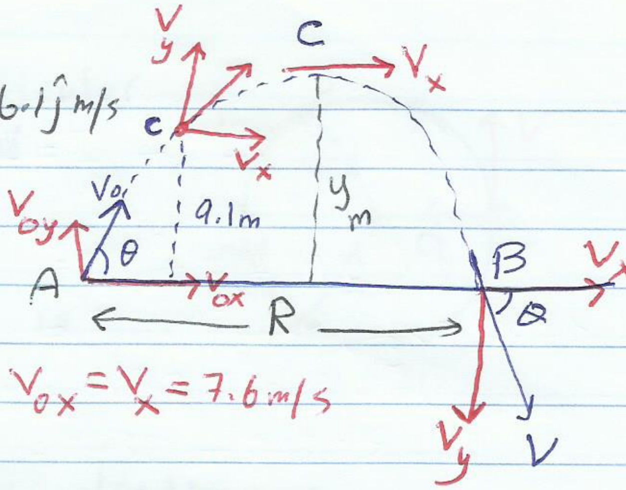
$$\vec{V} = 162\hat{i} - 170.5\hat{j} \ m/s$$

(4-43) at $y = 9.1m$ $\vec{V} = 7.6\hat{i} + 6.1\hat{j}$ m/s

find a) y_{max}

2) Range ?

3) find V_B ?



a) From $A \longrightarrow C$

the vertical motion

$V_{oy} \longrightarrow V_y = 6.1 m/s \longrightarrow \Delta y = 9.1m \longrightarrow a_y = -9.8 m/s^2$

$V_y^2 = V_{oy}^2 + 2a_y \Delta y$

$(6.1)^2 = V_{oy}^2 + 2(-9.8)(9.1)$, $V_{oy}^2 = (6.1)^2 + 2(9.8)(9.1)$
$= 37.2 + 178.4$

$V_{oy} = 14.7$ m/s

From $A \longrightarrow C$ (The vertical Motion)

$V_{oy} = 14.7 m/s \longrightarrow V_y = 0 \longrightarrow a_y = -9.8 m/s^2$

$V_y^2 = V_{oy}^2 + 2a_y \Delta y$

$0 = (14.7)^2 + 2(-9.8) y_m$

$y_m = \frac{(14.7)^2}{19.6} = 11 m$

c) Not: $V_x = 7.6 m/s$ stays constant from $A \longrightarrow B$

To find the time of flight $(A \longrightarrow B)$ y-Motion

$\Delta y = V_{oy} t + \frac{1}{2} a_y t^2$, $\Delta y = 0$ from $A \to B$

$0 = (14.7 m/s) t + \frac{1}{2}(-9.8) t^2$

$t = \frac{14.7}{4.9} = 3s$ $\boxed{t = 3s}$

Range $= V_{ox} t = (7.6)(3) = 22.8m$

c) $\vec{V_0} = 7.6\hat{i} + 14.7\hat{j}$ m/s , $\vec{V_B} = 7.6\hat{i} - 14.7\hat{j}$ m/s

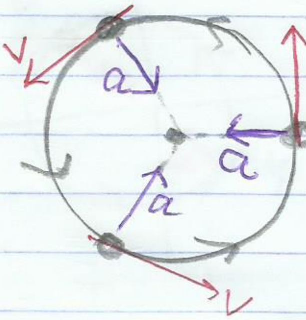
$\theta = \tan^{-1}\left(\frac{14.7}{7.6}\right) = 63°$

Uniform Circular Motion:

a particle is in Uniform circular motion if it travels around a circle or a circular arc

① at constant speed.

② the direction of the velocity is keeping changes



$V = \frac{2\pi r}{T}$, T: periodic time

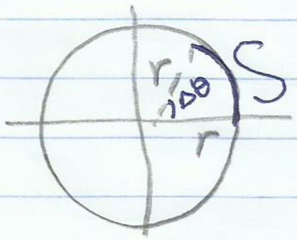
$V = \frac{2\pi r}{T}$ constant in magnitude

$a = \frac{V^2}{r}$ toward the center
centripetal acceleration

$f = \frac{1}{T}$ Hz (frequency) , $f = \frac{1}{T}$ (rev./s.)

$\omega = \frac{2\pi}{T} = 2\pi f$ angular speed

$\omega = \frac{2\pi}{T} = 2\pi f$ rad/s



In general:

the arc $= r\theta$

$S = r\theta$

$\frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$, ω = angular speed rad/s

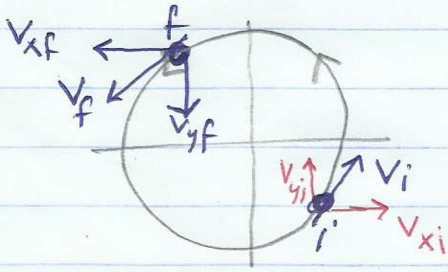
$\boxed{V = \omega r}$

Sample Problem (4.05) Top gun pilots in turns

$\vec{V}_i = (400\hat{i} + 500\hat{j})$ m/s

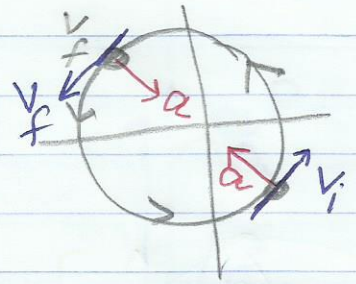
$\downarrow$ 24s $= t_{i\to f}$

$\vec{V}_f = (-400\hat{i} - 500\hat{j})$ m/s



Find a?

$V_i = V_f$ Uniform Circular Motion



$V_i = \sqrt{(400)^2 + (500)^2}$
$= 640\, m/s$

$V_f = \sqrt{(-400)^2 + (-500)^2}$

$= 640\, m/s$

the motion from (initial) $\xrightarrow{t=24s}$ final
is half period

$T = 2t = 48s.$

$V = \frac{2\pi R}{T}$, $R = \frac{VT}{2\pi} = \frac{(640)(48)}{2(3.14)}$

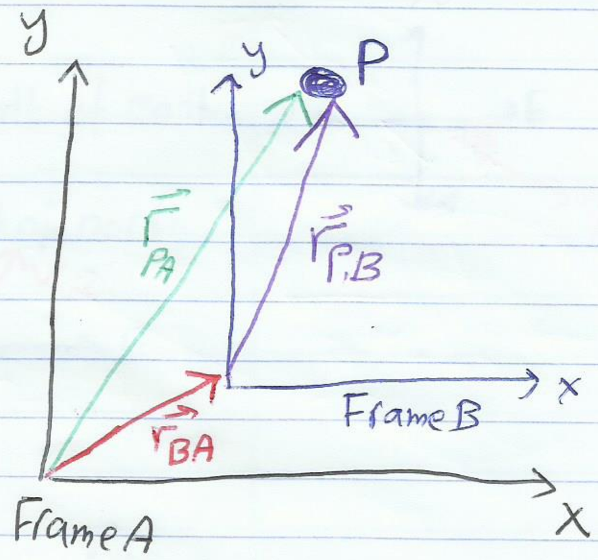
$R = 4889\, m$

$a_c = \frac{V^2}{R} = \frac{(640)^2}{4889} = 83.78\, m/s^2$

$a_c = \frac{83.78}{9.8} = 8.5g$

We will solve Problems
4-58 "

Relative Motion :



Frame A is At rest
Fram B is moving at
velocity $\vec{V}_{BA}$ is constant

both frames are observing
Point (P) :

$\vec{r}_{PA}$ = the position of (P)
relative to frame A

$\vec{r}_{PB}$ = the position of (P) relative to frame B

$\vec{r}_{BA}$ = the position of frame B relative to frame A

From Geometry $\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$

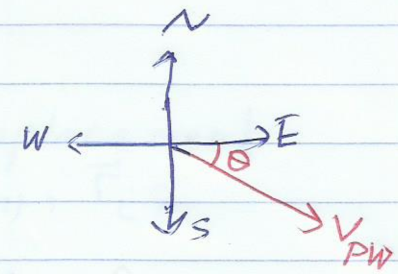
by taking the derivative of the equation relative to time

$$\vec{V}_{PA} = \vec{V}_{PB} + \vec{V}_{BA} \quad ,$$

take the derivative of the equation relative to time:

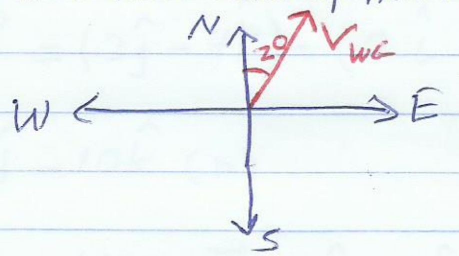
$$\vec{a}_{PA} = \vec{a}_{PB} \quad , \quad \frac{dV_{BA}}{dt} = 0$$

Sample Problem 4.08:



$\vec{V}_{PW} = 215 km/h$; directed at θ south of east

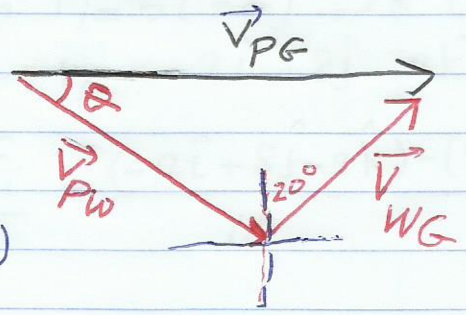
$\vec{V}_{WG} = 65 km/h$, directed $20°$ east of north



$\vec{V}_{PG}$, moves East

$|\vec{V}_{PG}|$?

$$\vec{V}_{PG} = \vec{V}_{PW} + \vec{V}_{WG}$$



$$V_{PG}\hat{i} = (215\cos\theta\,\hat{i} - 215\sin\theta\,\hat{j})$$
$$+$$
$$(65\sin 20)\hat{i} + (65\cos 20)\hat{j}$$

$$V_{PG}\hat{i} = (215\cos\theta + 65\sin 20)\hat{i} + (-215\sin\theta + 65\cos 20)\hat{j}$$

The x-components

$$V_{PG} = 215\cos\theta + 65\sin 20 \quad (1)$$

The y-components

$$0 = -215\sin\theta + 65\cos 20 \quad (2)$$

$$215\sin\theta = 65\cos 20$$

$$\sin\theta = \frac{65\cos 20}{215} = 0.284093 \quad \boxed{\theta = 16.5°}$$

equation (1) is:

$$V_{PG} = 215\cos 16.5 + 65\sin 20$$
$$= 228\ km/h$$

Chapter 4: Lecture Problems

(4-3) $\vec{\Delta r} = 2.0\hat{i} - 3.0\hat{j} + 6.0\hat{k}$ (m) Position's displacement
the final position is $\vec{r_f} = 3\hat{j} - 4\hat{k}$ (m), $\vec{r_i}$?

$\vec{\Delta r} = \vec{r_f} - \vec{r_i}$

$\vec{r_i} = \vec{r_f} - \vec{\Delta r} = (3\hat{j} - 4\hat{k}) - (2\hat{i} - 3\hat{j} + 6\hat{k})$

$\vec{r_i} = -2\hat{i} + 6\hat{j} - 10\hat{k}$ (m)

(4-7) ion's initial position $\vec{r_i} = 5\hat{i} - 6\hat{j} + 2\hat{k}$ (m)
After $\Delta t = 10s \longrightarrow \vec{r_f} = -2\hat{i} + 8\hat{j} - 2\hat{k}$ (m)

$$\vec{V}_{avg} = \frac{\vec{\Delta r}}{\Delta t} = \frac{\vec{r_f} - \vec{r_i}}{\Delta t} = \frac{(-2\hat{i} + 8\hat{j} - 2\hat{k}) - (5\hat{i} - 6\hat{j} + 2\hat{k})}{10}$$

$$\vec{V}_{avg} = \frac{-7\hat{i} + 14\hat{j} - 4\hat{k}}{10}$$

$$\vec{V}_{avg} = -0.7\hat{i} + 1.4\hat{j} - 0.4\hat{k} \text{ m/s}$$

(4-13) $\vec{r}(t) = \hat{i} + 4t^2\hat{j} + t\hat{k} \leftarrow$ Particle's position as a function of time

a) $\vec{V}(t) = \frac{d\vec{r}}{dt} = 8t\hat{j} + \hat{k}$ m/s

b) $\vec{a}(t) = \frac{d\vec{V}}{dt} = 8$ m/s^2

(4-15) A particle at $t = 0$; $\vec{r_0} = 0$, $\vec{V_0} = 3.00\hat{i}$ m/s
moves at constant $\vec{a} = (-1.00\hat{i} - 0.500\hat{j})$ m/s^2
When it reaches maximux $X \Rightarrow$ What are its $\vec{V}$? and $\vec{r}$?
At maximam X, $V_x = 0$

$V_x = V_{0x} + a_x t$ (Constant $\vec{a}$)

$0 = 3 + (-1)t \Longrightarrow \boxed{t = 3s}$ to reach maximum X

$V_y = V_{0y} + a_y t \Rightarrow V_y = 0 + -0.5(3) = -1.5$ m/s

$\vec{V} = -1.5\hat{j}$ m/s, $\vec{r} = \vec{r_0} + \vec{V_0}t + \frac{1}{2}\vec{a}t^2 = 0 + (3\hat{i})(3) + \frac{1}{2}(-1\hat{i} - 0.5\hat{j})(3$

$\vec{r} =$

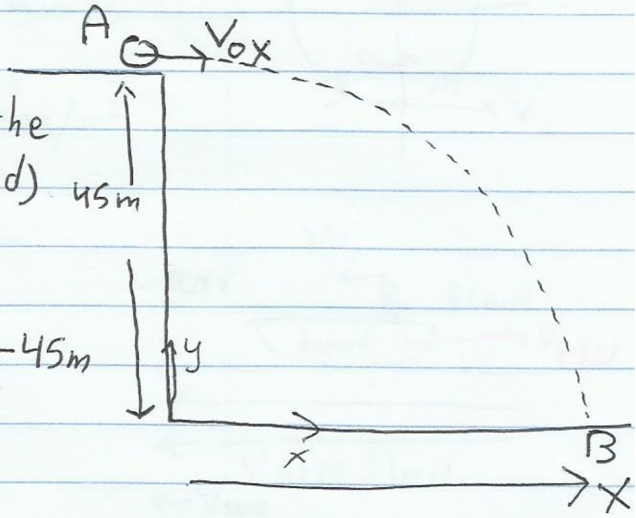
$\vec{r} = 9\hat{i} + -4.5\hat{i} + -2.25\hat{j}$ m

$\vec{r} = 4.5\hat{i} - 2.25\hat{j}$ m (Note that $x_{max} = 4.5$ m)

(4-23) $V_{ox} = 250$ m/s , $a_x = 0$

$V_{oy} = 0$, $a_y = -10$ m/s^2

$y_o = +45$ m $\longrightarrow y_f = 0$ (on the ground)



a) find t from $A \rightarrow B$)

from y-motion)

$\Delta y = V_{oy} t + \frac{1}{2} a_y t^2$, $\Delta y = y_f - y_o = -45$m

$-45\text{m} = 0 + \frac{1}{2}(-10) t^2$

$-45 = -5t^2$, $t = \sqrt{9} = 3$s

b) find X ? from $A \rightarrow B$.

$\Delta X = V_{ox} t = 250(3) = 750$m , $X_o = 0$ at A.

$X = 750$m.

c) find V_y ? at B ?

$V_y = V_{oy} + a_y t$ ($A \rightarrow B$)

$V_y = 0 + -10(3) = -30$ m/s direction

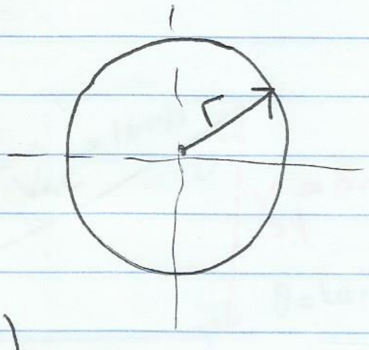
the magnitud of $V_y = 30$ m/s

Find the velocity of the projectile as it strikes the ground?

$\vec{V}_B = 250\hat{i} - 30\hat{j}$ m/s

(4-58) 1200 rev/min , $r = 0.15$m

frequency $= \frac{1200\text{rev}}{60\text{s}}$

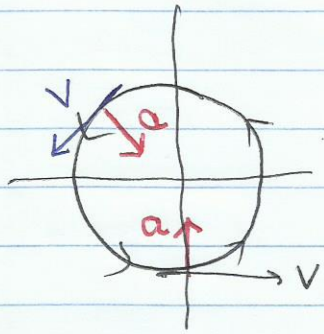


d) $f = 20$ rev/s $= 20$ Hz

$T = \frac{1}{20} = 0.05$ s (Periodic time)

a) In one rev. the tip. moves $2\pi r$

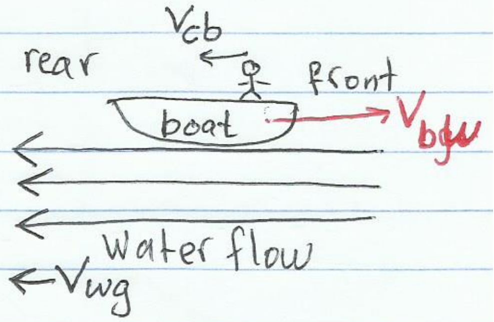
$C = 2\pi r = 0.942\,m$



b) $\text{speed} = \frac{C}{T} = \frac{0.942}{0.05} = 18.8\,m/s$

c) $a_c = \frac{V^2}{r} = \frac{(18.8)^2}{0.15} = 2.37 \times 10^3\,m/s^2$

(4-70) $V_{bw} = +14\hat{i}$ km/h



a)

$V_{wg} = -9\hat{i}$ km/h

$\vec{V}_{bg}$? $\vec{V}_{bg} = \vec{V}_{bw} + \vec{V}_{wg}$

Velocity of boat with respect to frame at rest (ground)

Velocity of boat relative to moving frame (water)

velocity of moving frame (w) relative to frame at rest (ground)

$\vec{V}_{bg} = +14\hat{i} + ^{-}9\hat{i} = {}^{+}5\hat{i}$ km/h

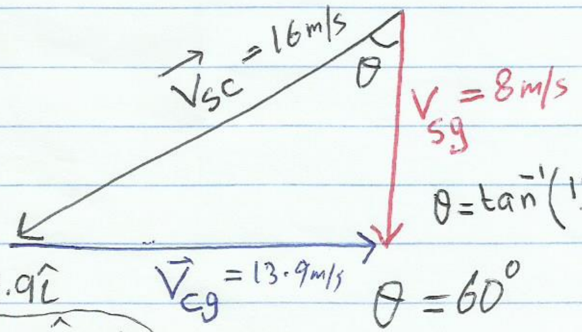
b) $\vec{V}_{cb} = -6\hat{i}$ km/h find $\vec{V}_{cg}$? c = child.

$\vec{V}_{cg} = \vec{V}_{cb} + \vec{V}_{bg}$

$\vec{V}_{cg} = -6\hat{i} + {}^{+}5\hat{i} = -1\hat{i}$ km/s

(4-77) $\vec{V}_{cg} = +50\hat{i}$ km/h $= 13.9\hat{i}$ m/s

$\vec{V}_{sg} = 8(-\hat{k})$ m/s $= -8\hat{k}$ m/s



$\theta = \tan^{-1}\left(\frac{13.9}{8}\right)$

$\theta = 60^0$

Note: $\vec{V}_{sg} = \vec{V}_{sc} + \vec{V}_{cg}$, $-8\hat{k} = \vec{V}_{sc} + 13.9\hat{i}$

$\vec{V}_{sc} = -13.9\hat{i} - 8\hat{k}$ m/s