8.7 Improper Integral

Def The definite integral with infinite limit of Integration are called improper integral type I

Case 1: If $f(x)$ is cont on $[a,\infty)$, then $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

Case 2: If $f(x)$ is cont on $(-\infty, b]$, then $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

Case 3: If $f(x)$ is cont on $(-\infty,\infty)$, then $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

$$= \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

Def. In each of the above cases, if the limit is finite, we say that the improper integral Converges

" Limite = The value of improper integral "

and if the value of the limit fails to exist, the improper integral is called diverges

* Remark

Con $\pm$ Conv. = Conv.

Conv. $\pm$ Div. = Div.

Div $\pm$ Div. = may be Conv. or Div.

Example Determine whether the following improper integral Converges or Diverges

① $\displaystyle\int_0^{\infty} \frac{16 \tan^{-1} x}{1+x^2}\, dx = 16 \lim_{b\to\infty} \int_0^{b} \frac{\tan^{-1} x}{1+x^2}\, dx$

$\displaystyle = 16 \lim_{b\to\infty} \left.\frac{(\tan^{-1}x)^2}{2}\right]_0^b$

$\displaystyle = 8 \lim_{b\to\infty} \left[(\tan^{-1}b)^2 - \cancel{(\tan^{-1}(0))^2}\right]$

$\displaystyle = 8\left(\frac{\pi}{2}\right)^2$

$= 2\pi^2$

$U = \tan^{-1} x$

$dU = \dfrac{1}{1+x^2}\, dx$

$\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\, dx$

$\displaystyle\int U\, dU = \frac{U^2}{2}$

$= \frac{1}{2}(\tan^{-1} x)^2$

Then $\displaystyle\int_0^{\infty} \frac{16 \tan^{-1} x}{1+x^2}\, dx$ Converges to $2\pi^2$

② $\displaystyle\int_2^{\infty} \frac{2}{x^2 - x}\, dx = 2 \lim_{b\to\infty} \int_2^{b} \frac{1}{x^2 - x}\, dx$

$\displaystyle = 2 \lim_{b\to\infty} \left[\ln\left|1-\frac{1}{x}\right|\right]_2^b$

$\displaystyle = 2 \lim_{b\to\infty} \ln\left|1-\frac{1}{b}\right| - \ln\left|\frac{1}{2}\right|$

$= 2\left[0 + \ln 2\right] = \ln 4$

So the improper integral Converges

$\displaystyle\int \frac{1}{x^2\left(1-\frac{1}{x}\right)}\, dx$

let $U = 1 - \frac{1}{x}$

$dU = \frac{1}{x^2}\, dx$

$\displaystyle\int \frac{dU}{U} = \ln|U|$

$= \ln\left|1-\frac{1}{x}\right|$

③ $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{a\to-\infty} \int_a^{-1} (x-1)^{-1/3}\,dx$

$$= \lim_{a\to-\infty} \left.\frac{(x-1)^{2/3}}{2/3}\right]_a^{-1}$$

$$= \frac{3}{2}\left[\lim_{a\to-\infty} \sqrt[3]{4} - (a-1)^{2/3}\right]$$

$$= -\infty$$

The limit is infinite, then $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x-1}}\,dx$ Diverges

④ $\displaystyle\int_{-\infty}^{\infty} \operatorname{sech} x\,dx = \int_{-\infty}^{\infty} \frac{2}{e^x+e^{-x}}\,dx$

$$= 2\int_0^{\infty} \frac{2}{e^x+e^{-x}}\,dx$$

$$= 2\lim_{b\to\infty} \int_0^b \frac{2e^x}{e^x(e^x+e^{-x})}\,dx$$

$$= 4\lim_{b\to\infty} \int_0^b \frac{e^x}{e^{2x}+1}\,dx$$

$$= 4\lim_{b\to\infty} \left.\tan^{-1}(e^x)\right]_0^b$$

$$= 4\lim_{b\to\infty} \left[\tan^{-1}(e^b) - \tan^{-1}(1)\right]$$

$$= 4\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \pi$$

$u = e^x$

$du = e^x\,dx$

$\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

$\displaystyle\int \frac{du}{u^2+1} = \tan^{-1}(u)$

$= \tan^{-1}(e^x)$

Conclusion

The limit is finite

then $\displaystyle\int_{-\infty}^{\infty} \operatorname{sech} x\,dx$ Converges

Theorem

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{Converge to } \dfrac{1}{p-1} & \text{if } p>1 \\ \text{Div.} & p \leq 1 \end{cases}$$

General Theorem

If $a>0$, then $\displaystyle\int_a^{\infty} \frac{dx}{x^p} = \begin{cases} \text{Converges} & \text{if } p>1 \\ \text{Diverges} & \text{if } p \leq 1 \end{cases}$

Def Integrals of function that become infinite at a point within the interval of integration are improper integral of Type II.

case [1] If $f(x)$ cont. on $(a,b]$ and discont at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

case [2] If $f(x)$ Cont. on $[a,b)$ and discont. at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

case [3] If $f(x)$ discont. at c, where $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx .$$

Def In each of a bove cases :-

* If the limit is finite we say that the improper integral converges "The limit is the value of the Improper Integral"

* If the limit does not exist, the integral Diverges

Exp: Evaluate the improper integral

$\boxed{1}$ $\displaystyle\int_0^1 \frac{4x}{\sqrt{1-x^4}}\,dx = \lim_{b\to 1^-} 2\int_0^b \frac{2x}{\sqrt{1-(x^2)^2}}\,dx$

$\displaystyle = 2\lim_{b\to 1^-} \sin^{-1}(x^2)\Big]_0^b$

$\displaystyle = 2\left[\lim_{b\to 1^-} \sin^{-1}(b^2) - \cancel{\sin^{-1}(0)}\right]$

$\displaystyle = 2\left[\frac{\pi}{2}\right] = \pi$

$\displaystyle\int \frac{2x\,dx}{\sqrt{1-(x^2)^2}}$

$u = x^2$

$du = 2x\,dx$

$\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) = \sin^{-1}(x^2)$

The limit is finite, Then the integral $\displaystyle\int_0^1 \frac{4x}{\sqrt{1-x^4}}\,dx$ converges.

$\boxed{2}$ $$\int_2^4 \frac{dx}{(x-2)^2} = \lim_{t \to 2^+} \int_t^4 \frac{dx}{(x-2)^2}$$

$$= \lim_{t \to 2^+} -\left[\frac{1}{x-2}\right]_t^4$$

$$= -\lim_{t \to 2^+} \left[\frac{1}{2} - \frac{1}{t-2}\right]$$

$$= -\left[\frac{1}{2} - \frac{1}{\text{small}+}\right] = \infty$$

The limit is infinite, then the integral Diverges.

$\boxed{3}$ $$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_{-1}^1 \frac{dy}{\sqrt{|y|}} = 2\int_0^1 \frac{dy}{\sqrt{|y|}}$$

since $\frac{1}{\sqrt{|y|}}$ even

Assume $y = x-1$ $\quad x=0 \longrightarrow y=-1$

$dy = dx$ $\quad x=2 \longrightarrow y=1$

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = 2\int_0^1 \frac{dy}{\sqrt{y}}$$

$$= 2\lim_{t \to 0^+} \int_t^1 \frac{dy}{\sqrt{y}}$$

$$= 2\lim_{t \to 0^+} \left.\frac{y^{1/2}}{\frac{1}{2}}\right]_t^1$$

$$= 4\lim_{t \to 0^+} \left[\sqrt{1} - \sqrt{t}\right] = 4$$

The limit is finite, Then the integral $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ converges

④ $$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

$$\int_0^1 \frac{dx}{(x-1)^2} = \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx = \lim_{b\to 1^-} -\left[\frac{1}{x-1}\right]_0^b$$

$$= -\lim_{b\to 1^-}\left[\frac{1}{b-1} - \frac{1}{-1}\right]$$

$$= -\left[\frac{1}{\text{small}-} + 1\right] = \infty$$

$$\int_1^2 \frac{dx}{(x-1)^2} = \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx = \lim_{b\to 1^+} -\left[\frac{1}{x-1}\right]_b^2$$

$$= -\lim_{b\to 1^+}\left[\frac{1}{1} - \frac{1}{b-1}\right]$$

$$= -\lim_{b\to 1^+}\left[1 - \frac{1}{\text{small}+}\right] = \infty$$

In this case $\int_0^2 \frac{dx}{(x-1)^2}$ Diverges (because limit $= 2\infty$
$= \infty$)

Test of Convergence of Improper integral

Direct Comparison Test :

let f and g be continuous on $[a, \infty)$

with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

[1] $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

[2] $\int_a^\infty g(x)\,dx$ Diverges if $\int_a^\infty f(x)\,dx$ Diverges

Exp1. Test the convergance and divergance of the following improper integral

[1] $\int_1^\infty e^{-x^2}\,dx$

$$e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{e^x}$$

$$\int_1^\infty \frac{1}{e^x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$-\lim_{b\to\infty} e^{-x}\Big]_1^b$$

$$\lim_{b\to\infty} -\left[e^{-b} - e^{-1}\right]$$

$$= \frac{1}{e}$$

But $\int_1^\infty \frac{1}{e^x}\,dx$ Converges

Then By D.C.T $\int_1^\infty e^{-x^2}\,dx$ conv.

$\boxed{2}$ $\displaystyle\int_1^\infty e^{-x} \cos^2 x$

$$\frac{\cos^2 x}{e^x} \leq \frac{1}{e^x}$$

But $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx$ conv. by previous example

By D.C.T. $\displaystyle\int_1^\infty e^{-x} \cos^2 x \,dx$ Conv.

$\boxed{3}$ $\displaystyle\int_1^\infty \frac{x \quad dx}{\sqrt{4x^5+1}}$ $\qquad \dfrac{x}{\sqrt{4x^5+1}} < \dfrac{x}{\sqrt{4x^5}} = \dfrac{x}{2x^{5/2}} = \dfrac{1}{2x^{3/2}}$

But $\displaystyle\int_1^\infty \frac{1}{2x^{3/2}}\,dx$ conv. ~~by~~ because $p=\frac{3}{2}>1$

Then $\displaystyle\int_1^\infty \frac{x}{\sqrt{4x^5+1}}\,dx$ Conv. by D.C.T

$\boxed{4}$ $\displaystyle\int_1^\infty \frac{2+\cos x}{x}\,dx$ $\qquad \dfrac{2+\cos x}{x} \geq \dfrac{1}{x}$

But $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ Div. because $p=1$

Then $\displaystyle\int_1^\infty \frac{2+\cos x}{x}\,dx$ Conv. by D.C.T.

$\boxed{5}$ $\displaystyle\int_3^{\infty} \frac{\sqrt{2x^2+7}}{x^3}\,dx$

$$\frac{\sqrt{2x^2+7}}{x^3} < \frac{\sqrt{2x^2+7x^2}}{x^3} = \frac{3x}{x^3} = \frac{3}{x^2}$$

But $\displaystyle\int_3^{\infty} \frac{3}{x^2}$ conv. because $p=2>1$

Then $\displaystyle\int_3^{\infty} \frac{\sqrt{2x^2+7}}{x^3}$ conv. by D.C.T

$\boxed{6}$ $\displaystyle\int_{100}^{\infty} \frac{x-99}{\sqrt{x^5+x+1}}\,dx$

$$\frac{x-99}{\sqrt{x^5+x+1}} < \frac{x}{\sqrt{x^5}} = \frac{1}{x^{3/2}}$$

But $\displaystyle\int_{100}^{\infty} \frac{1}{x^{3/2}}$ conv. because $p=\frac{3}{2}$

Then $\displaystyle\int_{100}^{\infty} \frac{x-99}{\sqrt{x^5+x+1}}\,dx$ conv. by D.C.T.

limit Comparison Test :-

If the positive function f and g are continuous on $[a,\infty)$

and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \quad , \quad 0 < L < \infty$$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both Converges or both Diverges

Example :- Test the convergence of the following integral

$\boxed{1}$ $\int_4^\infty \frac{2\,dx}{x^{3/2}-1}$

$\int_4^\infty \frac{dx}{x^{3/2}}$ Converges because $p=\frac{3}{2}>1$

$$\lim_{x\to\infty} \frac{2}{x^{3/2}-1} \Big/ \frac{1}{x^{3/2}} = \lim_{x\to\infty} \frac{2x^{3/2}}{x^{3/2}-1} = \frac{2}{1-\frac{1}{x^{3/2}}} = 2$$

By the L.C.T. $\int_4^\infty \frac{2\,dx}{x^{3/2}-1}$ Converge,

$\boxed{2}$ $\int_1^\infty \frac{dx}{\sqrt{3x+1}}$

$\int_1^\infty \frac{dx}{x^{1/2}}$ Div. because $P=1/2<1$

$$\lim_{x\to\infty} \frac{1}{\sqrt{3x+1}} \Big/ \frac{1}{\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{3x+1}} = \lim_{x\to\infty}\sqrt{\frac{x}{3x+1}} = \sqrt{\lim_{x\to\infty} \frac{x/x}{3x/x+1/x}}$$

$$= \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

By using L.C.T $\int_1^{\infty} \frac{dx}{\sqrt{3x+1}}$ div.

Another solution

$$\frac{1}{\sqrt{3x+1}} \geq \frac{1}{\sqrt{3x+x}} = \frac{1}{2\, x^{1/2}}$$

By using D.C.T $\int_1^{\infty} \frac{dx}{\sqrt{3x+1}}$ Div., because $\int_1^{\infty} \frac{1}{x^{1/2}} dx$ Div.

$\boxed{3}$ $\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx$

$\int_1^{\infty} \frac{1}{x^2} dx$ conv. because $P = 2 > 1$

$$\lim_{x \to \infty} \frac{\tan^{-1} x}{x^2} \Big/ \frac{1}{x^2} dx = \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

By using l.C.T $\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx$ Conv.

By using Another solution

$$\frac{\tan^{-1} x}{x^2} < \frac{\frac{\pi}{2}}{x^2}$$

and $\frac{\pi}{2} \int_1^{\infty} \frac{1}{x^2} dx$ conv. because $P=2$.

By using D.C.T. $\int \frac{\tan^{-1} x}{x^2}$ Conv.

(4) $\int_1^\infty \frac{dx}{\sqrt{x^2 - x+1}}$

$\int_1^\infty \frac{dx}{x}$ Div. because $P=1$

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2-x+1}} \Big/ \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{x/x}{\sqrt{\frac{x^2}{x^2}-\frac{x}{x^2}+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}} = 1$$

By L.C.T $\int_1^\infty \frac{dx}{\sqrt{x^2-x+2}}$ Div.

Another solution $\int_1^\infty \frac{dx}{\sqrt{x^2-x+1}}$

$$\int_1^\infty \frac{1}{\sqrt{x^2-x+1}} \geq \frac{1}{\sqrt{x^2-x+x}} = \frac{1}{\sqrt{x^2}} = \int_1^\infty \frac{1}{x}$$

By D.C.T $\int_1^\infty \frac{dx}{\sqrt{x^2-x+1}}$ Div.

$\boxed{5}$ $\displaystyle\int_1^\infty \frac{1}{e^x - 2^x}\,dx$

$$\int_1^\infty \frac{1}{e^x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -e^{-x}\Big]_1^b$$

$$= -\lim_{b\to\infty}\left[e^{-b} - \frac{1}{e}\right]$$

$$= \frac{1}{e} \quad \text{Conv.}$$

$$\lim_{x\to\infty} \frac{1}{e^x - 2^x} \Big/ \frac{1}{e^x}$$

$$= \lim_{x\to\infty} \frac{e^x/e^x}{\frac{e^x}{e^x} - \frac{2^x}{e^x}}$$

$$= \lim_{x\to\infty} \frac{1}{1 - \cancel{\left(\frac{2}{e}\right)^x}} = 1$$

By L.C.T $\displaystyle\int_1^\infty \frac{1}{e^x - 2^x}\,dx$ Conv.