Topic #5

16.30/31 Feedback Control Systems

State-Space Systems

- What are state-space models?
- Why should we use them?
- How are they related to the transfer functions used in classical control design and how do we develop a state-space model?
- What are the basic properties of a state-space model, and how do we analyze these?

SS Introduction

- State space model: a representation of the dynamics of an N^{th} order system as a first order differential equation in an N-vector, which is called the **state**.
  - Convert the N^{th} order differential equation that governs the dynamics into N first-order differential equations

- Classic example: second order mass-spring system

$$m\ddot{p} + c\dot{p} + kp = F$$

  - Let $x_1 = p$, then $x_2 = \dot{p} = \dot{x}_1$, and

$$\begin{aligned} \dot{x}_2 = \ddot{p} &= (F - c\dot{p} - kp)/m \\ &= (F - cx_2 - kx_1)/m \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

  - Let $u = F$ and introduce the state

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \Rightarrow \dot{\mathbf{x}} = A\mathbf{x} + Bu$$

  - If the measured output of the system is the position, then we have that

$$y = p = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C\mathbf{x}$$

- Most general continuous-time linear dynamical system has form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)\\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)\end{aligned}$$

where:

- $t \in \mathbb{R}$ denotes time
- $\mathbf{x}(t) \in \mathbb{R}^n$ is the state (vector)
- $\mathbf{u}(t) \in \mathbb{R}^m$ is the input or control
- $\mathbf{y}(t) \in \mathbb{R}^p$ is the output

- $A(t) \in \mathbb{R}^{n\times n}$ is the dynamics matrix
- $B(t) \in \mathbb{R}^{n\times m}$ is the input matrix
- $C(t) \in \mathbb{R}^{p\times n}$ is the output or sensor matrix
- $D(t) \in \mathbb{R}^{p\times m}$ is the feedthrough matrix

- Note that the plant dynamics can be time-varying.
- Also note that this is a multi-input / multi-output (MIMO) system.

- We will typically deal with the time-invariant case
$\Rightarrow$ **Linear Time-Invariant (LTI)** state dynamics

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t)\end{aligned}$$

so that now A, B, C, D are constant and do not depend on t.

Basic Definitions

- **Linearity** – What is a linear dynamical system? A system G is linear with respect to its inputs and output

$$\mathbf{u}(t) \rightarrow \boxed{G(s)} \rightarrow \mathbf{y}(t)$$

iff superposition holds:

$$G(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2) = \alpha_1 G\mathbf{u}_1 + \alpha_2 G\mathbf{u}_2$$

So if $\mathbf{y}_1$ is the response of G to $\mathbf{u}_1$ ($\mathbf{y}_1 = G\mathbf{u}_1$), and $\mathbf{y}_2$ is the response of G to $\mathbf{u}_2$ ($\mathbf{y}_2 = G\mathbf{u}_2$), then the response to $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$ is $\alpha_1\mathbf{y}_1 + \alpha_2\mathbf{y}_2$

- A system is said to be **time-invariant** if the relationship between the input and output is independent of time. So if the response to $\mathbf{u}(t)$ is $\mathbf{y}(t)$, then the response to $\mathbf{u}(t - t_0)$ is $\mathbf{y}(t - t_0)$

- Example: the system

$$\begin{aligned} \dot{x}(t) &= 3x(t) + u(t) \\ y(t) &= x(t) \end{aligned}$$

is LTI, but

$$\begin{aligned} \dot{x}(t) &= 3t\ x(t) + u(t) \\ y(t) &= x(t) \end{aligned}$$

is not.

  - A matrix of second system is a function of absolute time, so response to $u(t)$ will differ from response to $u(t-1)$.

- $\mathbf{x}(t)$ is called the **state of the system** at t because:
  - Future output depends only on current state and future input
  - Future output depends on past input only through current state
  - State summarizes effect of past inputs on future output – like the *memory of the system*

- **Example:** Rechargeable flashlight – the state is the *current state of charge* of the battery. If you know that state, then you do not need to know how that level of charge was achieved (assuming a perfect battery) to predict the future performance of the flashlight.
  - But to consider all nonlinear effects, you might also need to know how many cycles the battery has gone through
  - Key point is that you might expect a given linear model to accurately model the charge depletion behavior for a given number of cycles, but that model would typically change with the number cycles

Creating State-Space Models

- Most easily created from N^{th} order differential equations that describe the dynamics
  - This was the case done before.
  - Only issue is which set of states to use – there are many choices.

- Can be developed from transfer function model as well.
  - Much more on this later

- Problem is that we have restricted ourselves here to linear state space models, and almost all systems are nonlinear in real-life.
  - Can develop linear models from nonlinear system dynamics

Equilibrium Points

- Often have a nonlinear set of dynamics given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

where $\mathbf{x}$ is once gain the state vector, $\mathbf{u}$ is the vector of inputs, and $\mathbf{f}(\cdot, \cdot)$ is a nonlinear vector function that describes the dynamics

- First step is to define the point about which the linearization will be performed.
  - Typically about **equilibrium points** – a point for which if the system starts there it will remain there for all future time.

- Characterized by setting the state derivative to zero:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) = 0$$

  - Result is an algebraic set of equations that must be solved for both $\mathbf{x}_e$ and $\mathbf{u}_e$
  - Note that $\dot{\mathbf{x}}_e = 0$ and $\dot{\mathbf{u}}_e = 0$ by definition
  - Typically think of these nominal conditions $\mathbf{x}_e$, $\mathbf{u}_e$ as "set points" or "operating points" for the nonlinear system.

- Example – pendulum dynamics: $\ddot{\theta} + r\dot{\theta} + \frac{g}{l}\sin\theta = 0$ can be written in state space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -rx_2 - \frac{g}{l}\sin x_1 \end{bmatrix}$$

  - Setting $\mathbf{f}(\mathbf{x}, \mathbf{u}) = 0$ yields $x_2 = 0$ and $x_2 = -\frac{g}{rl}\sin x_1$, which implies that $x_1 = \theta = \{0, \pi\}$

Linearization

- Typically assume that the system is operating about some nominal state solution $\mathbf{x}_e$ (possibly requires a nominal input $\mathbf{u}_e$)
  - Then write the actual state as $\mathbf{x}(t) = \mathbf{x}_e + \delta\mathbf{x}(t)$ and the actual inputs as $\mathbf{u}(t) = \mathbf{u}_e + \delta\mathbf{u}(t)$
  - The "δ" is included to denote the fact that we expect the variations about the nominal to be "small"

- Can then develop the linearized equations by using the **Taylor series expansion** of $\mathbf{f}(\cdot, \cdot)$ about $\mathbf{x}_e$ and $\mathbf{u}_e$.

- Recall the vector equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, each equation of which

$$\dot{x}_i = f_i(\mathbf{x}, \mathbf{u})$$

can be expanded as

$$\begin{aligned} \frac{d}{dt}(x_{ei} + \delta x_i) &= f_i(\mathbf{x}_e + \delta\mathbf{x}, \mathbf{u}_e + \delta\mathbf{u}) \\ &\approx f_i(\mathbf{x}_e, \mathbf{u}_e) + \left.\frac{\partial f_i}{\partial \mathbf{x}}\right|_0 \delta\mathbf{x} + \left.\frac{\partial f_i}{\partial \mathbf{u}}\right|_0 \delta\mathbf{u} \end{aligned}$$

where

$$\frac{\partial f_i}{\partial \mathbf{x}} = \left[\begin{array}{ccc} \frac{\partial f_i}{\partial x_1} & \cdots & \frac{\partial f_i}{\partial x_n} \end{array} \right]$$

and $\cdot|_0$ means that we should evaluate the function at the nominal values of $\mathbf{x}_e$ and $\mathbf{u}_e$.

- The meaning of "small" deviations now clear – the variations in $\delta\mathbf{x}$ and $\delta\mathbf{u}$ must be small enough that we can ignore the higher order terms in the Taylor expansion of $\mathbf{f}(\mathbf{x}, \mathbf{u})$.

- Since $\frac{d}{dt}x_{ei} = f_i(\mathbf{x}_e, \mathbf{u}_e)$, we thus have that

$$\frac{d}{dt}(\delta x_i) \approx \left.\frac{\partial f_i}{\partial \mathbf{x}}\right|_0 \delta\mathbf{x} + \left.\frac{\partial f_i}{\partial \mathbf{u}}\right|_0 \delta\mathbf{u}$$

- Combining for all n state equations, gives (note that we also set "$\approx$" $\rightarrow$ "$=$") that

$$\begin{aligned}
\frac{d}{dt}\delta\mathbf{x} &= \begin{bmatrix} \left.\frac{\partial f_1}{\partial \mathbf{x}}\right|_0 \\ \left.\frac{\partial f_2}{\partial \mathbf{x}}\right|_0 \\ \vdots \\ \left.\frac{\partial f_n}{\partial \mathbf{x}}\right|_0 \end{bmatrix} \delta\mathbf{x} + \begin{bmatrix} \left.\frac{\partial f_1}{\partial \mathbf{u}}\right|_0 \\ \left.\frac{\partial f_2}{\partial \mathbf{u}}\right|_0 \\ \vdots \\ \left.\frac{\partial f_n}{\partial \mathbf{u}}\right|_0 \end{bmatrix} \delta\mathbf{u} \\
&= A(t)\delta\mathbf{x} + B(t)\delta\mathbf{u}
\end{aligned}$$

where

$$A(t) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_0 \quad \text{and} \quad B(t) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ & & \vdots & \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_0$$

- Similarly, if the nonlinear measurement equation is $\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$ and $\mathbf{y}(t) = \mathbf{y}_e + \delta\mathbf{y}$, then

$$\begin{aligned}\delta\mathbf{y} &= \begin{bmatrix} \left.\frac{\partial g_1}{\partial \mathbf{x}}\right|_0 \\ \vdots \\ \left.\frac{\partial g_p}{\partial \mathbf{x}}\right|_0 \end{bmatrix} \delta\mathbf{x} + \begin{bmatrix} \left.\frac{\partial g_1}{\partial \mathbf{u}}\right|_0 \\ \vdots \\ \left.\frac{\partial g_p}{\partial \mathbf{u}}\right|_0 \end{bmatrix} \delta\mathbf{u} \\ &= C(t)\delta\mathbf{x} + D(t)\delta\mathbf{u}\end{aligned}$$

- Typically drop the "δ" as they are rather cumbersome, and (abusing notation) we write the state equations as:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)\end{aligned}$$

  which is of the same form as the previous linear models

- If the system is operating around just one set point then the partial fractions in the expressions for A–D are all constant $\rightarrow$ **LTI linearized model.**

Stability of LTI Systems

- Consider a solution $\mathbf{x}_s(t)$ to a differential equation for a given initial condition $\mathbf{x}_s(t_0)$.
  - Solution is **stable** if other solutions $\mathbf{x}_b(t_0)$ that start near $\mathbf{x}_s(t_0)$ stay close to $\mathbf{x}_s(t)$ $\forall\, t$ $\Rightarrow$ **stable in sense of Lyapunov** (SSL).
  - If other solutions are SSL, but the $\mathbf{x}_b(t)$ do not converge to $\mathbf{x}_s(t)$ $\Rightarrow$ solution is **neutrally stable**.
  - If other solutions are SSL and $\mathbf{x}_b(t) \to \mathbf{x}(t)$ as $t \to \infty$ $\Rightarrow$ solution is **asymptotically stable**.
  - A solution $\mathbf{x}_s(t)$ is **unstable** if it is not stable.

- Note that a linear (autonomous) system $\dot{\mathbf{x}} = A\mathbf{x}$ has an equilibrium point at $\mathbf{x}_e = 0$
  - This equilibrium point is **stable** if and only if all of the eigenvalues of A satisfy $\mathbb{R}\lambda_i(A) \leq 0$ and every eigenvalue with $\mathbb{R}\lambda_i(A) = 0$ has a Jordan block of order one.[1]
  - Thus the stability test for a linear system is the familiar one of determining if $\mathbb{R}\lambda_i(A) \leq 0$

- Somewhat surprisingly perhaps, we can also infer stability of the original nonlinear from the analysis of the linearized system model

[1] more on Jordan blocks on 6–??, but this basically means that these eigenvalues are not repeated.

- **Lyapunov's indirect method**[2] Let $\mathbf{x}_e = 0$ be an equilibrium point for the nonlinear autonomous system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$$

where $\mathbf{f}$ is continuously differentiable in a neighborhood of $\mathbf{x}_e$. Assume

$$A = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_e}$$

Then:

  - The origin is an asymptotically stable equilibrium point for the nonlinear system if $\mathbb{R}\lambda_i(A) < 0 \;\forall\; i$
  - The origin is unstable if $\mathbb{R}\lambda_i(A) > 0$ for any i

- Note that this doesn't say anything about the stability of the nonlinear system if the linear system is neutrally stable.

- A very powerful result that is the basis of all linear control theory.

[2] Much more on Lyapunov methods later too.

Linearization Example

- **Example:** simple spring. With a mass at the end of a linear spring (rate k) we have the dynamics

$$m\ddot{x} = -kx$$

but with a "leaf spring" as is used on car suspensions, we have a nonlinear spring – the more it deflects, the stiffer it gets. Good model now is

$$m\ddot{x} = -k_1 x - k_2 x^3$$

which is a "cubic spring".

Fig. 1: Leaf spring from http://en.wikipedia.org/wiki/Image:Leafs1.jpg

- Restoring force depends on deflection x in a nonlinear way.

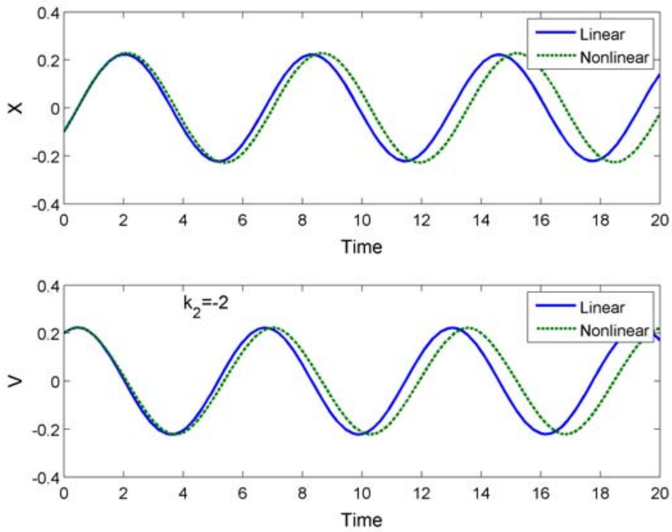

Fig. 2: Response to linear $k = 1$ and nonlinear ($k_1 = k, k_2 = -2$) springs (code at the end)

- Consider the nonlinear spring with (set $m = 1$)

$$\ddot{y} = -k_1 y - k_2 y^3$$

gives us the nonlinear model ($x_1 = y$ and $x_2 = \dot{y}$)

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -k_1 y - k_2 y^3 \end{bmatrix} \Rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

- Find the equilibrium points and then make a state space model

- For the equilibrium points, we must solve

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \dot{y} \\ -k_1 y - k_2 y^3 \end{bmatrix} = 0$$

which gives

$$\dot{y}_e = 0 \quad \text{and} \quad k_1 y_e + k_2 (y_e)^3 = 0$$

  - Second condition corresponds to $y_e = 0$ or $y_e = \pm\sqrt{-k_1/k_2}$, which is only real if k_1 and k_2 are opposite signs.

- For the state space model,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_0 = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 (y)^2 & 0 \end{bmatrix}_0$$

$$= \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 (y_e)^2 & 0 \end{bmatrix}$$

and the linearized model is $\dot{\delta\mathbf{x}} = A\delta\mathbf{x}$

- For the equilibrium point $y_e = 0$, $\dot{y}_e = 0$

$$A_0 = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix}$$

which are the standard dynamics of a system with **just** a linear spring of stiffness k_1

  - Stable motion about $y = 0$ if $k_1 > 0$

- Assume that $k_1 = -1$, $k_2 = 1/2$, then we should get an equilibrium point at $\dot{y} = 0$, $y = \pm\sqrt{2}$, and since $k_1 + k_2(y_e)^2 = 0$ then

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

which are the dynamics of a stable oscillator about the equilibrium point

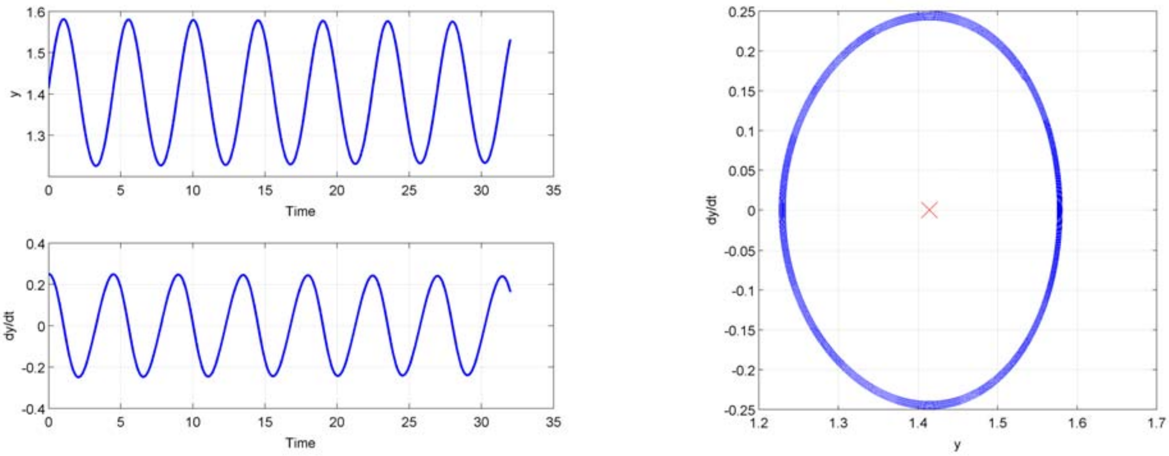

Fig. 3: Nonlinear response $(k_1 = -1, k_2 = 0.5)$. The figure on the right shows the oscillation about the equilibrium point.

Code: Nonlinear System Sim

```
function test=nlplant(ft);
global k1 k2
x0 = [-1 2]/10;

k1=1;k2=0;
[T,x]=ode23('plant', [0 20], x0); % linear

k2=-2;
[T1,x1]=ode23('plant', [0 20], x0);  %nonlinear

figure(1);clf;
subplot(211)
plot(T,x(:,1),T1,x1(:,1),'--');
legend('Linear','Nonlinear')
ylabel('X','FontSize',ft)
xlabel('Time','FontSize',ft)
subplot(212)
plot(T,x(:,2),T1,x1(:,2),'--');
legend('Linear','Nonlinear')
ylabel('V','FontSize',ft)
xlabel('Time','FontSize',ft)
text(4,0.3,['k_2=',num2str(k2)],'FontSize',ft)
return

% use the following to cll the function above
close all
set(0, 'DefaultAxesFontSize', 12, 'DefaultAxesFontWeight','demi')
set(0, 'DefaultTextFontSize', 12, 'DefaultTextFontWeight','demi')
set(0,'DefaultAxesFontName','arial')
set(0,'DefaultAxesFontSize',12)
set(0,'DefaultTextFontName','arial')
set(gcf,'DefaultLineLineWidth',2);
set(gcf,'DefaultlineMarkerSize',10)
global k1 k2
nlplant(14)
print -f1 -dpng -r300 nlplant.png
k1=-1;k2=0.5;
% call_plant.m
x0 = [sqrt(-k1/k2) .25];
[T,x]=ode23('plant', [0:.001:32], x0);
figure(1);subplot(211);plot(T,x(:,1));ylabel('y');xlabel('Time');grid
subplot(212);plot(T,x(:,2));ylabel('dy/dt');xlabel('Time');grid
figure(2);plot(x(:,1),x(:,2));grid
hold on;plot(x0(1),0,'rx','MarkerSize',20);hold off;
xlabel('y');ylabel('dy/dt')
axis([1.2 1.7 -.25 .25]);axis('square')

print -f1 -dpng -r300 nlplant2.png
print -f2 -dpng -r300 nlplant3.png
```

```
function [xdot] = plant(t,x);
% plant.m
global k1 k2
xdot(1) = x(2);
xdot(2) = -k1*x(1)-k2*(x(1))^3;
xdot = xdot';
```

Linearization Example: Aircraft Dynamics

- The basic dynamics are:

$$\vec{F} = m\,\dot{\vec{v}}_c^{\,I} \quad \text{and} \quad \vec{T} = \dot{\vec{H}}^I$$

$$\Rightarrow \frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + {}^{BI}\vec{\omega} \times \vec{v}_c \qquad \textbf{Transport Thm.}$$

$$\Rightarrow \vec{T} = \dot{\vec{H}}^B + {}^{BI}\vec{\omega} \times \vec{H}$$

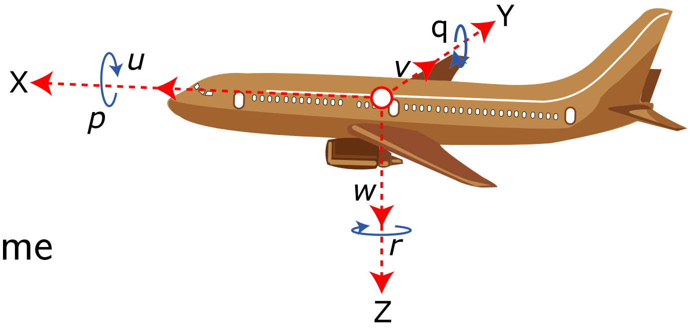

Image by MIT OpenCourseWare.

- Basic assumptions are:

  1. Earth is an inertial reference frame
  2. A/C is a rigid body
  3. Body frame **B** fixed to the aircraft $(\vec{i}, \vec{j}, \vec{k})$

- Instantaneous mapping of $\vec{v}_c$ and ${}^{BI}\vec{\omega}$ into the body frame:

$${}^{BI}\vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k} \qquad \vec{v}_c = U\vec{i} + V\vec{j} + W\vec{k}$$

$$\Rightarrow {}^{BI}\omega_B = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad \Rightarrow (v_c)_B = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

- If x and z axes in plane of symmetry, can show that $I_{xy} = I_{yz} = 0$, but value of I_{xz} depends on specific body frame selected.
  - Instantaneous mapping of angular momentum

$$\vec{H} = H_x\vec{i} + H_y\vec{j} + H_z\vec{k}$$

  into the body frame given by

$$H_B = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

- The overall equations of motion are then:

$$\frac{1}{m}\vec{F} = \dot{\vec{v}}_c^B + {}^{BI}\vec{\omega} \times \vec{v}_c$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$= \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

$$\vec{T} = \dot{\vec{H}}^B + {}^{BI}\vec{\omega} \times \vec{H}$$

$$\Rightarrow \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} \\ I_{yy}\dot{Q} \\ I_{zz}\dot{R} + I_{xz}\dot{P} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

- Equations are very nonlinear and complicated, and we have not even said where $\vec{F}$ and $\vec{T}$ come from $\Rightarrow$ need to linearize to develop analytic results
  - Assume that the aircraft is flying in an *equilibrium condition* and we will linearize the equations about this nominal flight condition.

Linearization

- Can linearize about various steady state conditions of flight.
  - For steady state flight conditions must have
    $$\vec{F} = \vec{F}_{\text{aero}} + \vec{F}_{\text{gravity}} + \vec{F}_{\text{thrust}} = 0 \quad \text{and } \vec{T} = 0$$
    * So for equilibrium condition, forces balance on the aircraft $L = W$ and $T = D$
  - Also assume that $\dot{P} = \dot{Q} = \dot{R} = \dot{U} = \dot{V} = \dot{W} = 0$
  - Impose additional constraints that depend on **flight condition:**
    * Steady wings-level flight $\rightarrow \Phi = \dot{\Phi} = \dot{\Theta} = \dot{\Psi} = 0$

- Define the **trim** angular rates and velocities

$$^{BI}\omega_B^o = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad (v_c)_B^o = \begin{bmatrix} U_o \\ 0 \\ 0 \end{bmatrix}$$

which are associated with the flight condition. In fact, these define the type of equilibrium motion that we linearize about. **Note:**

  - $W_0 = 0$ since we are using the stability axes, and
  - $V_0 = 0$ because we are assuming symmetric flight

- Proceed with linearization of the dynamics for various flight conditions

	Nominal Velocity	Perturbed Velocity	⇒	Perturbed Acceleration
Velocities	$U_0,$	$U = U_0 + u$	$\Rightarrow$	$\dot{U} = \dot{u}$
	$W_0 = 0,$	$W = w$	$\Rightarrow$	$\dot{W} = \dot{w}$
	$V_0 = 0,$	$V = v$	$\Rightarrow$	$\dot{V} = \dot{v}$
Angular Rates	$P_0 = 0,$	$P = p$	$\Rightarrow$	$\dot{P} = \dot{p}$
	$Q_0 = 0,$	$Q = q$	$\Rightarrow$	$\dot{Q} = \dot{q}$
	$R_0 = 0,$	$R = r$	$\Rightarrow$	$\dot{R} = \dot{r}$
Angles	$\Theta_0,$	$\Theta = \Theta_0 + \theta$	$\Rightarrow$	$\dot{\Theta} = \dot{\theta}$
	$\Phi_0 = 0,$	$\Phi = \phi$	$\Rightarrow$	$\dot{\Phi} = \dot{\phi}$
	$\Psi_0 = 0,$	$\Psi = \psi$	$\Rightarrow$	$\dot{\Psi} = \dot{\psi}$

- **Linearization for symmetric flight**

  $U = U_0 + u$, $V_0 = W_0 = 0$, $P_0 = Q_0 = R_0 = 0$.

  Note that the forces and moments are also perturbed.

$$\frac{1}{m}[X_0 + \Delta X] = \dot{U} + QW - RV \approx \dot{u} + qw - rv \approx \dot{u}$$

$$\frac{1}{m}[Y_0 + \Delta Y] = \dot{V} + RU - PW \approx \dot{v} + r(U_0 + u) - pw \approx \dot{v} + rU_0$$

$$\frac{1}{m}[Z_0 + \Delta Z] = \dot{W} + PV - QU \approx \dot{w} + pv - q(U_0 + u) \approx \dot{w} - qU_0$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} + rU_0 \\ \dot{w} - qU_0 \end{bmatrix} \quad \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

- Attitude motion:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{yy}\dot{q} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{matrix}$$

- To understand equations in detail, and the resulting impact on the vehicle dynamics, we must investigate terms $\Delta X \ldots \Delta N$.
  - We must also address the left-hand side ($\vec{F}$, $\vec{T}$)
  - **Net** forces and moments must be zero in equilibrium condition.
  - Aerodynamic and Gravity forces are a function of equilibrium condition **AND** the perturbations about this equilibrium.

- Predict the changes to the aerodynamic forces and moments using a first order expansion in the key flight parameters

$$\begin{aligned}\Delta X &= \frac{\partial X}{\partial U}\Delta U + \frac{\partial X}{\partial W}\Delta W + \frac{\partial X}{\partial \dot{W}}\Delta \dot{W} + \frac{\partial X}{\partial \Theta}\Delta\Theta + \ldots + \frac{\partial X^g}{\partial \Theta}\Delta\Theta + \Delta X^c \\ &= \frac{\partial X}{\partial U}u + \frac{\partial X}{\partial W}w + \frac{\partial X}{\partial \dot{W}}\dot{w} + \frac{\partial X}{\partial \Theta}\theta + \ldots + \frac{\partial X^g}{\partial \Theta}\theta + \Delta X^c\end{aligned}$$

- $\frac{\partial X}{\partial U}$ called **stability derivative** – evaluated at eq. condition.
- Clearly approximation since ignores lags in aerodynamics forces (assumes that forces only function of instantaneous values)

Stability Derivatives

- First proposed by Bryan (1911) – has proven to be a **very** effective way to analyze the aircraft flight mechanics – well supported by numerous flight test comparisons.
- The forces and torques acting on the aircraft are very complex nonlinear functions of the flight equilibrium condition and the perturbations from equilibrium.
  - Linearized expansion can involve many terms $u, \dot{u}, \ddot{u}, \ldots, w, \dot{w}, \ddot{w}, \ldots$
  - Typically only retain a few terms to capture the dominant effects.
- Dominant behavior most easily discussed in terms of the:
  - Symm. variables: U, W, Q & forces/torques: X, Z, and M
  - Asymm. variables: V, P, R & forces/torques: Y, L, and N
- Observation – for truly symmetric flight Y, L, and N will be exactly **zero** for any value of U, W, Q

  $\Rightarrow$ Derivatives of asymmetric forces/torques with respect to the symmetric motion variables are **zero**.

- Further (convenient) assumptions:
  1. Derivatives of symmetric forces/torques with respect to the asymmetric motion variables are small and can be neglected.
  2. We can neglect derivatives with respect to the derivatives of the motion variables, but keep $\partial Z/\partial \dot{w}$ and $M_{\dot{w}} \equiv \partial M/\partial \dot{w}$ (aerodynamic lag involved in forming new pressure distribution on the wing in response to the perturbed angle of attack)
  3. $\partial X/\partial q$ is negligibly small.

$\partial()/\partial()$	X	Y	Z	L	M	N
u	•	0	•	0	•	0
v	0	•	0	•	0	•
w	•	0	•	0	•	0
p	0	•	0	•	0	•
q	≈ 0	0	•	0	•	0
r	0	•	0	•	0	•

- Note that we must also find the perturbation gravity and thrust forces and moments

$$\left.\frac{\partial X^g}{\partial \Theta}\right|_0 = -mg\cos\Theta_0 \qquad \left.\frac{\partial Z^g}{\partial \Theta}\right|_0 = -mg\sin\Theta_0$$

- **Aerodynamic summary:**

1A $\Delta X = \left(\frac{\partial X}{\partial U}\right)_0 u + \left(\frac{\partial X}{\partial W}\right)_0 w \Rightarrow \Delta X \sim u,\ \alpha_x \approx w/U_0$

2A $\Delta Y \sim \beta \approx v/U_0,\ p,\ r$

3A $\Delta Z \sim u,\ \alpha_x \approx w/U_0,\ \dot{\alpha}_x \approx \dot{w}/U_0,\ q$

4A $\Delta L \sim \beta \approx v/U_0,\ p,\ r$

5A $\Delta M \sim u,\ \alpha_x \approx w/U_0,\ \dot{\alpha}_x \approx \dot{w}/U_0,\ q$

6A $\Delta N \sim \beta \approx v/U_0,\ p,\ r$

- Result is that, with these force, torque approximations, equations **1**, **3**, **5** decouple from **2**, **4**, **6**

- **1**, **3**, **5** are the **longitudinal dynamics** in u, w, and q

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} = \begin{bmatrix} m\dot{u} \\ m(\dot{w} - qU_0) \\ I_{yy}\dot{q} \end{bmatrix}$$

$$\approx \begin{bmatrix} \left(\frac{\partial X}{\partial U}\right)_0 u + \left(\frac{\partial X}{\partial W}\right)_0 w + \left(\frac{\partial X^g}{\partial \Theta}\right)_0 \theta + \Delta X^c \\ \left(\frac{\partial Z}{\partial U}\right)_0 u + \left(\frac{\partial Z}{\partial W}\right)_0 w + \left(\frac{\partial Z}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial Z}{\partial Q}\right)_0 q + \left(\frac{\partial Z^g}{\partial \Theta}\right)_0 \theta + \Delta Z^c \\ \left(\frac{\partial M}{\partial U}\right)_0 u + \left(\frac{\partial M}{\partial W}\right)_0 w + \left(\frac{\partial M}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial M}{\partial Q}\right)_0 q + \Delta M^c \end{bmatrix}$$

- **2**, **4**, **6** are the **lateral dynamics** in v, p, and r

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} m(\dot{v} + rU_0) \\ I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix}$$

$$\approx \begin{bmatrix} \left(\frac{\partial Y}{\partial V}\right)_0 v + \left(\frac{\partial Y}{\partial P}\right)_0 p + \left(\frac{\partial Y}{\partial R}\right)_0 r + +\Delta Y^c \\ \left(\frac{\partial L}{\partial V}\right)_0 v + \left(\frac{\partial L}{\partial P}\right)_0 p + \left(\frac{\partial L}{\partial R}\right)_0 r + \Delta L^c \\ \left(\frac{\partial N}{\partial V}\right)_0 v + \left(\frac{\partial N}{\partial P}\right)_0 p + \left(\frac{\partial N}{\partial R}\right)_0 r + \Delta N^c \end{bmatrix}$$

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