Birzeit University
Mathematics Department

Linear systems Math 330

$n \times n$ Linear system of equations $AX = b$:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}
$$

Augmented matrix: $[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$

Methods of solving $n \times n$ linear system:

(1) **Gaussian Elimination**: Transform $[A|b]$ to $[U|C]$ (upper triangular system), then solve (for X) by back substitution.

(2) **LU Factorization**: Write $A = LU$ ($\Longrightarrow LUX = b$), then solve $LY = b$ (for Y) by forward substitution, then solve $UX = Y$ (for X) by back substitution.

(3) **Gauss-Jordan Reduction**: Transform $[A|b]$ to $[I|X]$.

(4) **Inverse method**: Find A^{-1} by transforming $[A|I]$ to $[I|A^{-1}]$, then $X = A^{-1}b$

(5) **Cramer's Rule**: $x_i = \dfrac{|A_i|}{|A|}$ for $i = 1, 2, \cdots, n$

Gaussian Elimination

Ex: Solve the following system:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 6x_2 & + & 4x_3 & + & 3x_4 & = & 22 \\ x_1 & + & 2x_2 & + & x_3 & + & 4x_4 & = & 13 \\ 4x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 20 \\ -3x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 6 \end{array}$$

Solution:

First: Transform $[A|b]$ to $[U|C]$

$$\left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 1 & 2 & 1 & 4 & 13 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

Step 1: $m_{21} = \frac{1}{2} = 0.5$, $m_{31} = \frac{4}{2} = 2$, $m_{41} = \frac{-3}{2} = -1.5$

$$\begin{array}{l} \\ R_2 - 0.5R_1 \\ R_3 - 2R_1 \\ R_4 + 1.5R_1 \end{array} \left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 0 & -1 & -1 & 2.5 & 2 \\ 0 & -10 & -6 & -5 & -24 \\ 0 & 10 & 9 & 6.5 & 39 \end{array}\right]$$

Step 2: $m_{32} = \frac{-10}{-1} = 10$, $m_{42} = \frac{10}{-1} = -10$

$$\begin{array}{l} \\ \\ R_3 - 10R_2 \\ R_4 + 10R_2 \end{array} \left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 0 & -1 & -1 & 2.5 & 2 \\ 0 & 0 & 4 & -30 & -44 \\ 0 & 0 & -1 & 31.5 & 59 \end{array}\right]$$

Step 3: $m_{43} = \frac{-1}{4} = -0.25$

$$\begin{array}{l} \\ \\ \\ R_4 + 0.25R_3 \end{array} \left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 0 & -1 & -1 & 2.5 & 2 \\ 0 & 0 & 4 & -30 & -44 \\ 0 & 0 & 0 & 24 & 48 \end{array}\right]$$

Second: Use Back Substitution:

$$x_4 = 2 \ , \ x_3 = 4 \ , \ x_2 = -1 \ , \ x_1 = 3$$

$$\boxed{\text{Total cost} = 46 + 16 = 62}$$

LU Factorization

Ex: Solve the following system:

$$\begin{array}{rcrcrcrcl} 2x_1 & + & 6x_2 & + & 4x_3 & + & 3x_4 & = & 22 \\ x_1 & + & 2x_2 & + & x_3 & + & 4x_4 & = & 13 \\ 4x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 20 \\ -3x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 6 \end{array}$$

Solution:

First: Write $A = LU$

$$A = \begin{bmatrix} 2 & 6 & 4 & 3 \\ 1 & 2 & 1 & 4 \\ 4 & 2 & 2 & 1 \\ -3 & 1 & 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 4 & 3 \\ 0 & -1 & -1 & 2.5 \\ 0 & -10 & -6 & -5 \\ 0 & 10 & 9 & 6.5 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 4 & 3 \\ 0 & -1 & -1 & 2.5 \\ 0 & 0 & 4 & -30 \\ 0 & 0 & -1 & 31.5 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 4 & 3 \\ 0 & -1 & -1 & 2.5 \\ 0 & 0 & 4 & -30 \\ 0 & 0 & 0 & 24 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 6 & 4 & 3 \\ 0 & -1 & -1 & 2.5 \\ 0 & 0 & 4 & -30 \\ 0 & 0 & 0 & 24 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 2 & 10 & 1 & 0 \\ -1.5 & -10 & -0.25 & 1 \end{bmatrix}$$

Second: Solve $LY = b$ using forward substitution:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 2 & 10 & 1 & 0 \\ -1.5 & -10 & -0.25 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 22 \\ 13 \\ 20 \\ 6 \end{bmatrix}$$

$$y_1 = 22\ ,\ y_2 = 2,\ y_3 = -44\ ,\ y_4 = 48$$

Third: Solve $UX = Y$ using backward substitution:

$$\begin{bmatrix} 2 & 6 & 4 & 3 \\ 0 & -1 & -1 & 2.5 \\ 0 & 0 & 4 & -30 \\ 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 22 \\ 2 \\ -44 \\ 48 \end{bmatrix}$$

$$x_4 = 2\ ,\ x_3 = 4\ ,\ x_2 = -1\ ,\ x_1 = 3$$

$$\boxed{\text{Total cost} = 34 + 12 + 16 = 62}$$

Gauss-Jordan Reduction

Ex: Solve the following system:

$$\begin{array}{ccccccc} 2x_1 & + & 2x_2 & - & 4x_3 & = & 6 \\ x_1 & - & x_2 & + & x_3 & = & 0 \\ 4x_1 & + & x_2 & + & 2x_3 & = & 2 \end{array}$$

Solution: Transform $[A|b]$ to $[I|X]$

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 6 \\ 1 & -1 & 1 & 0 \\ 4 & 1 & 2 & 2 \end{array}\right]$$

Step 1:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 1 & -1 & 1 & 0 \\ 4 & 1 & 2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & -2 & 3 & -3 \\ 0 & -3 & 10 & -10 \end{array}\right]$$

Step 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & -1.5 & 1.5 \\ 0 & -3 & 10 & -10 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -0.5 & 1.5 \\ 0 & 1 & -1.5 & 1.5 \\ 0 & 0 & 5.5 & -5.5 \end{array}\right]$$

Step 3:

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.5 & 1.5 \\ 0 & 1 & -1.5 & 1.5 \\ 0 & 0 & 1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\Longrightarrow \quad x_1 = 1\ ,\ x_2 = 0\ ,\ x_3 = -1$$

$\boxed{\text{Total cost} = 30}$

Inverse Method

Ex: Solve the following system:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & x_3 & = & 7 \\ -x_1 & - & 2x_2 & + & 2x_3 & = & 1 \\ 4x_1 & + & 3x_2 & - & x_3 & = & 7 \end{array}$$

Solution:

First: Find A^{-1} by transforming $[A|I]$ to $[I|A^{-1}]$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ -1 & -2 & 2 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow\longrightarrow\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1.75 & -1.5 & -1.25 \\ 0 & 0 & 1 & 1.25 & -0.5 & -0.75 \end{array}\right]$$

Second: $X = A^{-1}b$

$$X = \begin{bmatrix} -1 & 1 & 1 \\ 1.75 & -1.5 & -1.25 \\ 1.25 & -0.5 & -0.75 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_1 = 1 \ , \ x_2 = 2 \ , \ x_3 = 3$$

$$\boxed{\text{Total cost } = 60 + 15 = 75}$$

Cramer's Rule

Ex: Solve the following system:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & x_3 & = & 7 \\ -x_1 & - & 2x_2 & + & 2x_3 & = & 1 \\ 4x_1 & + & 3x_2 & - & x_3 & = & 7 \end{array}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -2 & 2 \\ 4 & 3 & -1 \end{bmatrix},\ A_1 = \begin{bmatrix} 7 & 1 & 1 \\ 1 & -2 & 2 \\ 7 & 3 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 & 7 & 1 \\ -1 & 1 & 2 \\ 4 & 7 & -1 \end{bmatrix},\ A_3 = \begin{bmatrix} 2 & 1 & 7 \\ -1 & -2 & 1 \\ 4 & 3 & 7 \end{bmatrix},$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{4}{4} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{8}{4} = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{12}{4} = 3$$

$$\boxed{\text{Total cost} = 4(14) + 3 = 59}$$