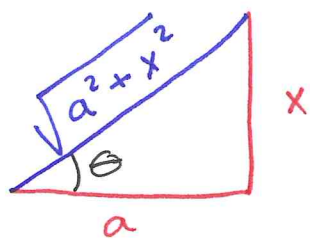


8-3 Trigonometric Substitution

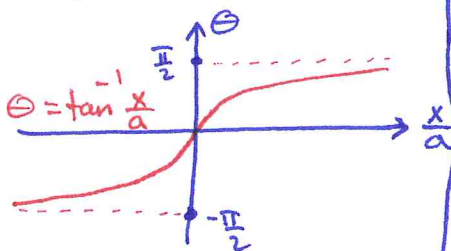
36



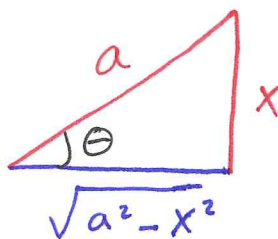
$$x = a \tan \theta \quad \text{requires}$$

$$\theta = \tan^{-1} \frac{x}{a} \quad \text{with}$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \heartsuit^1$$



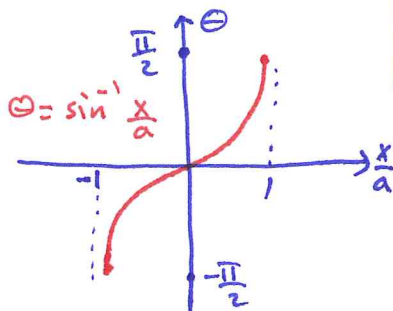
$$\begin{aligned} \sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2 \tan^2 \theta} \\ &= a \sqrt{1 + \tan^2 \theta} \\ &= a \sqrt{\sec^2 \theta} \\ &= a |\sec \theta| \quad \heartsuit^1 \\ &= a \sec \theta \end{aligned}$$



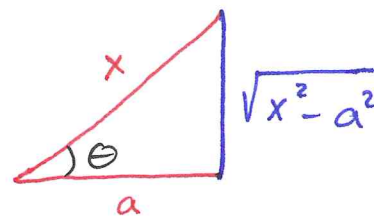
$$x = a \sin \theta \quad \text{requires}$$

$$\theta = \sin^{-1} \frac{x}{a} \quad \text{with}$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \heartsuit^2$$



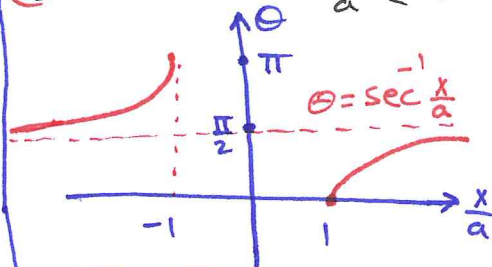
$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= a \sqrt{1 - \sin^2 \theta} \\ &= a \sqrt{\cos^2 \theta} \\ &= a |\cos \theta| \quad \heartsuit^2 \\ &= a \cos \theta \end{aligned}$$



$$x = a \sec \theta \quad \text{requires}$$

$$\theta = \sec^{-1} \frac{x}{a} \quad \text{with}$$

$$\begin{aligned} 0 \leq \theta < \frac{\pi}{2} &\quad \text{if } \frac{x}{a} \geq 1 \quad \heartsuit^3 \\ \frac{\pi}{2} < \theta \leq \pi &\quad \text{if } \frac{x}{a} \leq -1 \end{aligned}$$



$$\begin{aligned} \sqrt{x^2 - a^2} &= \sqrt{a^2 \sec^2 \theta - a^2} \\ &= a \sqrt{\sec^2 \theta - 1} \\ &= a \sqrt{\tan^2 \theta} \\ &= a |\tan \theta| \quad \heartsuit^3 \\ &= a \tan \theta \end{aligned}$$

Exp $\int \frac{dx}{\sqrt{9+x^2}}$

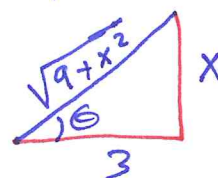
$$x = 3 \tan \theta \Rightarrow dx = 3 \sec^2 \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

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$$\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3 \sec^2 \theta d\theta}{\sqrt{9+9 \tan^2 \theta}} = \int \frac{\sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}} = \int \frac{\sec^2 \theta d\theta}{|\sec \theta|}$$

$$= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$

$$= \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| + C = \ln |\sqrt{9+x^2} + x| + C.$$



Exp $\int \sqrt{25-t^2} dt$ $t = 5 \sin \theta \Rightarrow dt = 5 \cos \theta d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (37)

$$\int \sqrt{25-t^2} dt = \int \sqrt{25-25\sin^2\theta} 5 \cos \theta d\theta$$

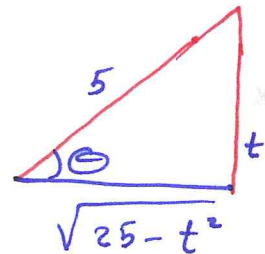
$$= 25 \int \sqrt{1-\sin^2\theta} \cos \theta d\theta = 25 \int \sqrt{\cos^2\theta} \cos \theta d\theta$$

$$= 25 \int \cos^2\theta d\theta = \frac{25}{2} \int (1 + \cos 2\theta) d\theta = \frac{25}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] + C$$

$$= \frac{25}{2} \left[\theta + \sin \theta \cos \theta \right] + C$$

$$= \frac{25}{2} \left[\sin^{-1}\left(\frac{t}{5}\right) + \frac{t}{5} \frac{\sqrt{25-t^2}}{5} \right] + C$$

$$= \frac{25}{2} \sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$$



Exp $\int \frac{\sqrt{y^2-25}}{y^3} dy$ $y = 5 \sec \theta \Rightarrow dy = 5 \sec \theta \tan \theta d\theta$
 $0 < \theta < \frac{\pi}{2}$

$$\int \frac{\sqrt{y^2-25}}{y^3} dy = \int \frac{\sqrt{25\sec^2\theta-25}}{125\sec^3\theta} 5 \sec \theta \tan \theta d\theta$$

$$= \frac{1}{5} \int \frac{\sqrt{\sec^2\theta-1}}{\sec^2\theta} \tan \theta d\theta = \frac{1}{5} \int \frac{\tan^2\theta}{\sec^2\theta} d\theta = \frac{1}{5} \int \sin^2\theta d\theta$$

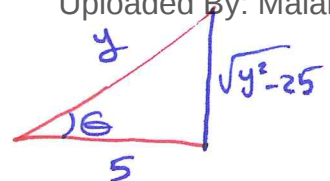
$$= \frac{1}{10} \int (1 - \cos 2\theta) d\theta = \frac{1}{10} \left[\theta - \frac{\sin 2\theta}{2} \right] + C$$

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$$= \frac{1}{10} \left[\theta - \sin \theta \cos \theta \right] + C$$

$$= \frac{1}{10} \left[\sec^{-1}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{y} \left(\frac{5}{y}\right) \right] + C$$

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Exp $\int_{-2}^2 \frac{dx}{4+x^2}$

$$x = 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta d\theta$$

$$\boxed{-\frac{\pi}{4} < \theta < \frac{\pi}{4}}$$

(38)

when $x = -2 \Rightarrow \theta = \tan^{-1}(-1) = -\frac{\pi}{4}$

$x = 2 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$\int_{-2}^2 \frac{dx}{4+x^2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2 \sec^2 \theta d\theta}{4 + 4 \tan^2 \theta} = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \tan^2 \theta} d\theta$$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta = \frac{1}{2} \left[\frac{\pi}{4} + \frac{\pi}{4} \right] = \frac{\pi}{4}$$

Note that $\int_{-2}^2 \frac{dx}{4+x^2} = \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_{-2}^2$

$$= \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} -1 \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - -\frac{\pi}{4} \right]$$

$$= \frac{\pi}{4}$$