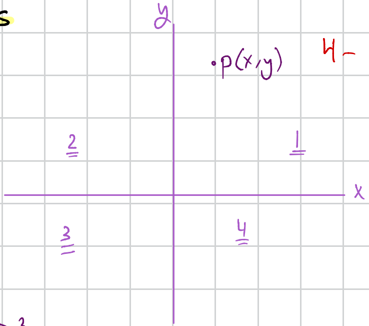
Wednesday بسم الله الرحمن الرحيم 18- sep - 2024

Chapter 12: Vectors and the geometry of space

12.1: three-dimensional coordinate systems

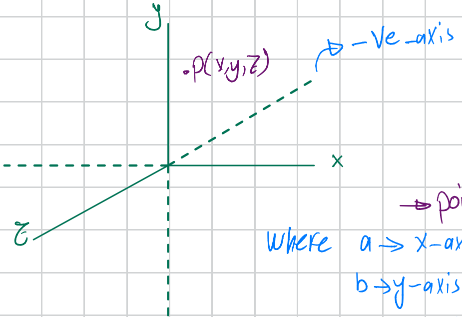
→ The point $p(a,b) \in \mathbb{R}^2$ $(\mathbb{R} \times \mathbb{R})$

هاي الطاقة لطولين plane هي بس y axis



4 - Quandrants

٤ ارباع



→ point $p(a,b,c) \in \mathbb{R}^3$

where $a \to x$-axis

$b \to y$-axis

$c \to z$-axis

and there are 8 Octans in space ٨ اثمان

Note:

لما بدي اعبر عن xy -plane
↳ z must equal zero

$\Rightarrow$ xy-plane → z = 0

xz-plane → y = 0

yz-plane → x = 0

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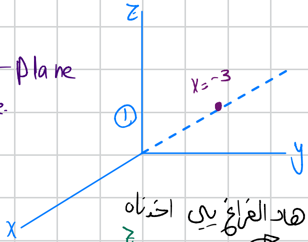
example: Geometric description in $\mathbb{R}^3$

① $x = -3$: The plane perpindicular to x-axis at $x = -3$ and Parallel to yz-plane

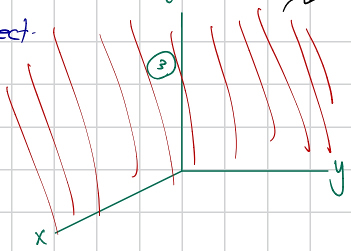
$y = 4$: The plane perpindicular to y-axis at $y = 4$ and parallel to xz-plane.



② $x = 2$ & $y = 3$: The line Pass through (2,3,0) and parallel to z-axis and ⊥ to xy-Plane

$y = -2$ & $z = 2$: The line in which the planes $y = -2$ & $z = 2$ intersect.

هاد الفراغ يلي اخدناه

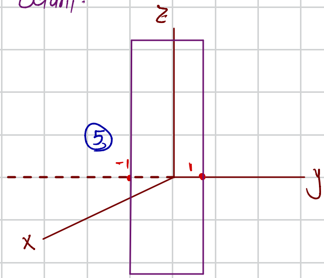


③ $z \geq 0$: The Half space consisting of the points That abdle xy-Plane

④ $x \geq 0$, $y \geq 0$, $z \geq 0$: all of the axis is +ve so the space we must take it is the first octant.

⑤ $-1 \geq y \geq 1$: the Slab is between $y = 1$ & $y = -1$

• هون ثبتنا قيم y و x & z هو ما
فبطلع لنا شريحة (Slab)



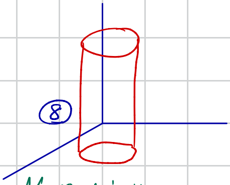
⑥ $x^2+y^2=4$ & $z=3$: Circle in the plane $z=3$ while the center is $(0,0,3)$ & Radius $=2$

xy-plane ⊥ z axis , in $\mathbb{R}^3$: $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Center (a,b,c) & Radius $=r$

⑦ $x^2+y^2\leq 1$ & $z=3$: Disk in the plane $z=3$ with Center $(0,0,3)$

⑧ $x^2+y^2\leq 1$: Cylindrical Solid with $r=1$ & z axis ⊥ center



⑨ $\underline{z=0}$, $x\leq 0$, $y\geq 0$: Second Quadrant in xy-plane

↳ xy-Plane

No restriction on z

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* Distance and Spheres in space :

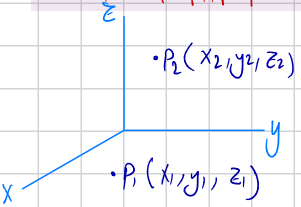
→ The Distance between $P_1(x_1,y_1,z_1)$ & $P_2(x_2,y_2,z_2)$

$$|P_2P_1|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$|P_2P_1|=|P_1P_2|$ كمسافة فقط

← لما نصير نعتبرهم Vectors

$|P_2P_1|\neq|P_1P_2|$



example : Find the distance between $p_1(1,1,1)$ & $p_2(2,3,5)$

$$|P_2P_1|=\sqrt{(2-1)^2+(3-1)^2+(5-1)^2}=\sqrt{21}$$

→ The standard equation for the sphere in Space

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$$ } equation of sphere with radius $=\sqrt{a^2}=a$ & Center $=(x_0,y_0,z_0)$

example : Find the center & Radius for " $x^2+y^2+z^2+3x-4z+1=0$ "

$$x^2+3x+\overset{2\,(\frac{3}{2})^2}{\frac{9}{4}}+y^2+z^2-4z+\overset{(\frac{4}{2})^2}{4}+1=0+\frac{9}{4}+4$$

جمعنا كل الحدود يلي نفس الشيء مع بعض & بدنا نكمل مربع

$\left(\frac{\text{معامل } x}{2}\right)^2$

$$\left(x+\frac{3}{2}\right)^2+(y+0)^2+(z-2)^2=\frac{9}{4}+3=\frac{21}{4}$$

∴ Center $\left(\frac{-3}{2},0,2\right)$ & radius $=\sqrt{\frac{21}{4}}$

$\Rightarrow (a+b)^2=a^2+2ab+b^2$

لما اكمل المربع بنتقل من

example: $\boxed{1}$ $x^2+y^2+z^2>4 \Rightarrow$ exterior of the sphere $x^2+y^2+z^2=4$

$\boxed{2}$ $x^2+y^2+z^2<4 \Rightarrow$ interior of the Sphere $x^2+y^2+z^2=4$

$\boxed{3}$ $x^2+y^2+z^2\leq 4 \Rightarrow$ Sphere with its interior

$\boxed{4}$ $x^2+y^2+z^2=4$ & $z\leq 0 \Rightarrow$ The lower hemisphere Cut from the sphere $x^2+y^2+z^2=4$ by xy-plane

$\boxed{5}$ $\underbrace{x^2+y^2=4}_{\text{cylinder}}$ & $\underbrace{z=y}_{\text{Plane}}$ $\Rightarrow$ (قطع ناقص) ellipse هاي بتعطيني اسطوانة لما اجي اقطعها بشكل مائل بنتج عندي

↳ The ellipse formed by the intersection of the cylinder $x^2+y^2=4$ & the plane $z=y$

32: the set of points in space equidistant from the $\underset{O(0,0,0)}{\text{origin}}$ and the point $\underset{Q}{(0,2,0)}$

$$|\overline{PO}| = |\overline{PQ}|$$

• بعد هاي النقطة عن origin = بعد هاي النقطة عن Q

$$\left(\sqrt{x^2+y^2+z^2} = \sqrt{x^2+(y-2)^2+z^2}\right)^2$$

$$\cancel{x^2}+y^2+\cancel{z^2} = \cancel{x^2}+(y-2)^2+\cancel{z^2}$$

$$\cancel{y^2} = \cancel{y^2}-4y+4 \Rightarrow 4y=4 \Rightarrow \boxed{y=1}$$

34: The Set of Point in Space that lie 2 units from the point $\overset{R}{(0,0,1)}$ at the Same time, 2 units From the point $(0,0,-1)$ Q

$|PR|=2$ & $|PQ|=2 \rightarrow \sqrt{x^2+y^2+(z+1)^2}=2 \rightarrow x^2+y^2+(z+1)^2=4$ — ②

$\Downarrow$

$\sqrt{x^2+y^2+(z-1)^2}=2 \rightarrow x^2+y^2+(z-1)^2=4$ — ①

• Let be $1=2$

$$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-2z+\cancel{1} = \cancel{x^2}+\cancel{y^2}+\cancel{z^2}+2z+\cancel{1}$$

$$-2z=2z \rightarrow 4z=0 \rightarrow z=0$$

• Substitute $z=0$ in equation 1

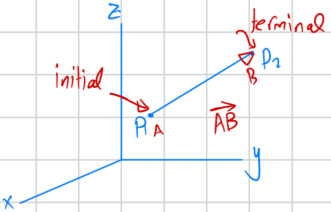
$x^2+y^2+(-1)^2=4$

$x^2+y^2=3$ & $z=0$

Circle in xy plane

Saturday 21-Sep-2024

12.2: vectors



Def: a Vector in adirected line Segment from initial point $p(x_1,y_1,z_1)$ and terminal point (x_2,y_2,z_2)

* we usually give names For vectors ⇒ $\vec{u}, \vec{v}$ or $\overset{\text{bold}}{\mathbf{u}, \mathbf{v}}$

* Vectors are equal if they have Same length & direction

* $\vec{v} = \langle v_1, v_2, v_3 \rangle$, point (x_1, y_1, z_1)

example: $p(-3,4,1)$ & $Q(-5,2,2)$ ⇒ Find the Components & direction of $\vec{PQ}$

∘ $\vec{PQ} = \langle -5-3 , 2-4 , 2-1 \rangle$
$= \langle -2, -2, 1 \rangle$

$|\vec{PQ}| = \sqrt{(-2)^2+(-2)^2+1^2} = 3$

∘ direction $= \frac{\vec{PQ}}{|\vec{PQ}|} = \langle \frac{-2}{3}, \frac{-2}{3}, \frac{1}{3} \rangle$

* let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be Vectors and k: scaler

. $\vec{u} + \vec{v} = \langle u_1+v_1 , u_2+v_2 , u_3+v_3 \rangle$

. $k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$

example: $\vec{u} = \langle -1, 3, 1 \rangle$, $\vec{v} = \langle 4, 7, 0 \rangle$

a) $\vec{u} + 3\vec{v} = \langle -1,3,1 \rangle + \langle 12, 21, 0 \rangle = \langle 11, 24, 1 \rangle$

b) $|\frac{1}{2}\vec{v}| =$ $\frac{1}{2}\vec{v} = \langle 2, \frac{7}{2}, 0 \rangle \Rightarrow |\frac{1}{2}\vec{v}| = \sqrt{2^2+(\frac{7}{2})^2+0^2} = \sqrt{4+\frac{49}{4}} = \sqrt{\frac{65}{4}} = \frac{\sqrt{65}}{2}$

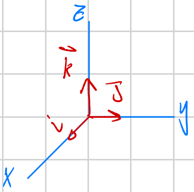
* properties of Vector operations:

1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3) $\vec{u} + 0 = \vec{u}$
4) $\vec{u} + (-\vec{u}) = \vec{0}$
5) $0\vec{u} = \vec{0}$
6) $1\vec{u} = \vec{u}$
7) $a(b\vec{u}) = (ab)\vec{u}$
8) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
9) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

* Unit Vector : A Vector $\vec{v}$ of length 1

→ The standard unit Vector are $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

$\vec{v} = (x,y,z) \Rightarrow \vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$



Remark: a unit Vector in the direction of $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$

a unit Vector in the opposite direction of $\vec{v}$ is $-\frac{\vec{v}}{|\vec{v}|}$

example: Find a vector with length 7 units in the direction of $\vec{v} = 12\vec{i} - 5\vec{k}$

vector $= \frac{7\vec{v}}{|\vec{v}|} = 7\left(\frac{12\vec{i} - 5\vec{k}}{\sqrt{144+25}}\right) = \frac{84\vec{i}}{13} - \frac{35\vec{k}}{13}$

example: $\vec{v} = 3\vec{i} - 4\vec{j}$ be a Velocity Vector, express $\vec{v}$ as a product of its Speed times a unit Vector in the direction of motion.

→ Speed is the Magnitude (length) of $\vec{v}$ given by $|\vec{v}| = \sqrt{9+16} = \sqrt{25} = 5$

→ The unit Vector $\frac{\vec{v}}{|\vec{v}|} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$ has the Same direction of $\vec{v}$

So $\vec{v} = 3\vec{i} - 4\vec{j} = \underbrace{5}_{\text{length (Speed)}}\underbrace{\left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right)}_{\text{Direction of motion.}}$

* Midpoint of a line segment :

The midpoint M of the line segment joining the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

example: find the midpoint of the Segment joining $P_1(3,-2,0)$ & $P_2(7,4,0)$

$$M = \left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+0}{2}\right) \Rightarrow M = (5,1,0)$$

⇒ The direction of $\overrightarrow{P_1P_2}$

$\vec{v} = \overrightarrow{P_1P_2} = (7-3)\vec{i} + (4+2)\vec{j} + (0-0)\vec{k}$

$\vec{v} = 4\vec{i} + 6\vec{j} + \vec{k}$ $\Rightarrow |\vec{v}| = \sqrt{16+36+0} = \sqrt{52} = 2\sqrt{13}$

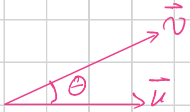
→ The direction is $\frac{\vec{v}}{|\vec{v}|} = \frac{4}{2\sqrt{13}}\vec{i} + \frac{6}{2\sqrt{13}}\vec{j} + \frac{1}{2\sqrt{13}}\vec{k}$

12.3: The Dot product

Def: the Dot product $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$ (Scalar) or $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

Where $\vec{u} = \langle u_1, u_2, u_3\rangle$ & $\vec{v} = \langle v_1, v_2, v_3\rangle$

- The angle between $\vec{u}$ & $\vec{v}$ (θ) is $\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$



Remark: • Two Vectors are orthogonal (perpendicular) if and only if $\vec{u}\cdot\vec{v} = 0$ $\left(\cos\frac{\pi}{2} = 0\right)$

• $\vec{0}$ is orthogonal to any Vector $\vec{u}$ since $\vec{0}\cdot\vec{u} = 0u_1 + 0u_2 + 0u_3 = 0$

example: * $\vec{u} = 3\vec{i} - 2\vec{j} + \vec{k}$, $\vec{v} = 2\vec{j} + 4\vec{k}$ are orthogonal

$\vec{u}\cdot\vec{v} = 3(0) + -2(2) + 1(4)$

$= 0 + -4 + 4 = 0$ #

* if $\vec{u} = 3\vec{i} - 2\vec{j} + \vec{k}$ & $\vec{v} = 2j + x\vec{k}$ are orthogonal, find x

$\vec{u}\cdot\vec{v} = 3(0) + (-2)(2) + 1(x) = 0$

$0 - 4 + x = 0 \Rightarrow \underline{\underline{x = 4}}$

* Find the angle between $\vec{u} = \vec{i} - 2\vec{j} - k$ & $\vec{v} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

$\theta = \cos^{-1}\left(\frac{6 + (-6) - 2}{\sqrt{1+4+1}\sqrt{36+9+4}}\right) = \cos^{-1}\left(\frac{-2}{\sqrt{6}\sqrt{49}}\right) \Rightarrow \underline{\underline{\theta = 96.69^\circ}}$

* Properties of Dot Product:

1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ 2) $(c\vec{u})\cdot\vec{v} = \vec{u}(c\vec{v}) = c(\vec{v}\cdot\vec{u})$ 3) $\vec{u}\cdot\vec{u} = |\vec{u}|^2$

4) $\vec{u}\cdot(\vec{v} + \vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ 5) $\vec{0}\cdot\vec{u} = 0$

example: $|\vec{u}| = 4$, $|\vec{v}| = 5$, $\theta = \frac{\pi}{3}$, find $|\vec{u} + \vec{v}|$?

$|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v})\cdot(\vec{u} + \vec{v})$ $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

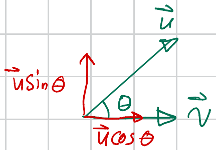
$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$

$= 16 + (4)(5)\cos\frac{\pi}{3} + (5)(4)\cos\frac{\pi}{3} + 25$

$|\vec{u} + \vec{v}|^2 = 61 \Rightarrow |\vec{u} + \vec{v}| = \sqrt{61}$

* Vector projection:

$\text{proj}_{\vec{v}} \vec{u}$ → projection for $\vec{u}$ at the Same direction of $\vec{v}$



$$\text{Proj}_{\vec{v}} \vec{u} = (\text{length})(\text{direction})$$

$$= (\vec{u}\cos\theta)\ \frac{\vec{v}}{|\vec{v}|}$$

← بنضرب وبنقسم على قيمة $\vec{v}$

مش دايما بعطينا قيمة θ

$$= |\vec{u}|\,|\vec{v}|\cos\theta\ \frac{\vec{v}}{|\vec{v}|}\ \frac{1}{|\vec{v}|}$$

$$= \boxed{\frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}} \cdot \frac{\vec{v}}{|\vec{v}|}$$

⇓ Scalar component. $= \text{Comp}_{\vec{v}} \vec{u}$

example: find $\text{Proj}_{\vec{v}} \vec{u}$ when $\vec{u} = 6i + 3j + 2k$, $\vec{v} = i - 2j - 2k$ and find $\text{Comp}_{\vec{v}} \vec{u}$

$\vec{u}\cdot\vec{v} = \cancel{6} - \cancel{6} - 4 = -4$

① $$\text{Proj}_{\vec{v}} \vec{u} = \frac{(\vec{u}\cdot\vec{v})}{\vec{v}} \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{-4}{3} \cdot \frac{i - 2j - 2k}{3}$$

$$= -\frac{4}{9}\vec{i} + \frac{8}{9}\vec{j} + \frac{8}{9}\vec{k}$$

② $$\text{Comp}_{\vec{v}} \vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}} = \frac{-4}{\sqrt{9}} = \frac{-4}{3}$$

12.4: The cross product

$$\underset{\text{Vector}}{\vec{u} \times \vec{v}} = \underbrace{\underset{\text{length}}{(|\vec{u}||\vec{v}| \sin\theta)}}_{\text{Scaler}} \underline{\underline{\vec{n}}} \rightarrow$$ is the unit Vector normal to the plane Containing $\vec{u}$ & $\vec{v}$

عمودي على plane ⇐ ن عمودي على $\vec{u}$ & $\vec{v}$

Remark: • non zero Vector $\vec{u}$ & $\vec{v}$ are parallel to each other iff $\vec{u} \times \vec{v} = 0$

• $\vec{u}$ & $\vec{v}$ are perpendicular (orthogonal) to each other iff $\vec{u} \cdot \vec{v} = 0$

• The cross product $\vec{u} \times \vec{v}$ is Zero Vector if $\theta = 0$ <u>or</u> $\theta = \pi$ <u>or</u> $\vec{u}$ is zero Vector <u>or</u> $\vec{v}$ is Zero Vector.

* properties of cross product:

1) $(r\vec{u})(s\vec{v}) = (rs)(\vec{u} \times \vec{v})$

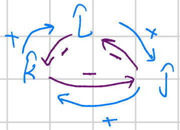
2) $\vec{u}(\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

3) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

4) $(\vec{v} + \vec{w})\vec{u} = (\vec{v} \times \vec{u}) + (\vec{w} \times \vec{u})$

5) $\vec{0} \times \vec{u} = \vec{0}$ (have direction)

6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$



$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$$|\vec{u} \times \vec{v}| = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i\,(u_2v_3 - v_2u_3) - j\,(u_1v_3 - v_1u_3) + k\,(u_1v_2 - v_1u_2)$$

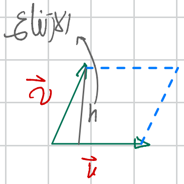
example: $\vec{u} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{v} = -4\vec{i} + 3\vec{j} + \vec{k}$

a) $$\vec{u} \times \vec{v} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = i(1-3) - j(2+4) + k(6+4)$$
$$= -2i - 6j + 10k$$

b) find a unit Vector normal to the plane Containing $\vec{u}$ & $\vec{v}$

$$\Rightarrow \text{unit Vector} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \frac{-2i - 6j + 10k}{\sqrt{4 + 36 + 100}} = \frac{-2\vec{i} - 6\vec{j} + 10k}{\sqrt{140}}$$

- $|\vec{u} \times \vec{v}|$ is the area of the parallelogran



$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta$

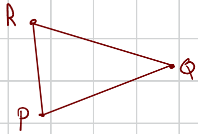
Area $= (\text{base})(\text{hight})$

$= |\vec{u}||\vec{v}| \sin\theta \, (h)$

$= \vec{u} \times \vec{v} \circ h$

* area of triangle $= \frac{1}{2} |\vec{u} \times \vec{v}|$

example: Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ & $R = (-1,1,2)$ and find the area of triangle ΔPQR



$\vec{PR} = \langle -2, 2, 2 \rangle$

$\vec{PQ} = \langle 1, 2, -1 \rangle$

a) $\vec{PR} \times \vec{PQ}$

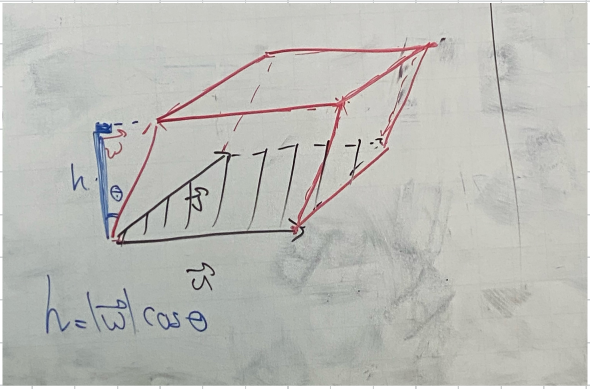
$$\vec{PR} \times \vec{PQ} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = i(-2-4) - j(\cancel{2-2}_{0}) + k(-4-2)$$

$= -6i - 6k$

b) Area $= \frac{1}{2} |\vec{PQ} \times PQ| = \frac{-6i - 6k}{2} = -3i - 3k$

* Triple Scaler or Box product:

$\Rightarrow$ The Volume $= (\text{Area of Base})(\text{hight})$



$h = |\vec{w}| \cos\theta$

$V = |\vec{u} \times \vec{v}| \, |\vec{w}| \cos\theta$

$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

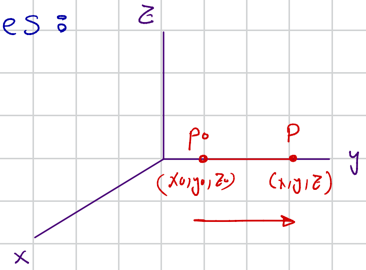
$$V = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

example: Find The Volume of The box determined by $\vec{u} = i + 2j + k$, $\vec{v} = -2i + 3k$, $\vec{w} = 7j - 4k$

$$V = \begin{vmatrix} \overset{+}{1} & \overset{-}{-2} & \overset{+}{1} \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = |-23| = 23$$

12.5: Lines and planes in space

Lines :



$L = \{P(x,y,z) : \overrightarrow{P_0P} \parallel \vec{v}\}$

$\overrightarrow{P_0P} = t\vec{v}$

$(x-x_0)i + (y-y_0)j + (z-z_0)k = t(v_1 i + v_2 j + v_3 k)$

$xi + yj + zk = (x_0 i + y_0 j + z_0 k) + t(v_1 i + v_2 j + v_3 k)$

1) معادلة الخط المستقيم في الفراغ ←

$\vec{r}(t) = r_0(t) + t\vec{v}$ → Velocity $|\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right)$

$\vec{r}$ ↓ position Vector

$r_0(t)$ ↓ initial position

t → Time

While $\vec{r}$ is the position vector.

2) parametric equations:

$x = x_0 + tv_1 \Rightarrow t = x - x_0 / v_1$

$y = y_0 + tv_2 \quad t = y - y_0 / v_2$

$z = z_0 + tv_3 \quad t = z - z_0 / v_3$

t is equal.

3) Symmetric equations:

$$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$$

example: ⓐ Find parametric equation for the line through $P(-3,2,-3)$ & $Q(1,-1,4)$

$P_0 = P = (-3, 2, -3)$

$\vec{v} = \overrightarrow{PQ} = 4i - 3j + 7k$

$x = -3 + 4t$

$y = 2 - 3t$

$z = -3 + 7t$

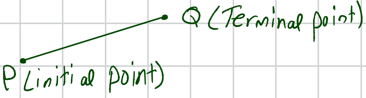
منقدر نختار Po يا إما (P or Q)

بطلع جوابين مختلفين بس both is true.

- Symmetric equations

$$\frac{x+3}{4} = \frac{y-2}{-3} = \frac{z+3}{7}$$

ⓑ Parametrize the line segment $\overrightarrow{PQ}$



$\vec{v} = \overrightarrow{PQ} = 4i - 3j + 7k$

$P_0 = P = (-3, 2, -3)$

→ to find The limit of t

$P(\overset{x_0}{-3}, \overset{y_0}{2}, \overset{z_0}{-3})$

$x = -3 + 4t$

$-3 = -3 + 4t \rightarrow t = 0$

$Q(\overset{x}{1}, \overset{y}{-1}, \overset{z}{4})$

$x = -3 + 4t$

$1 = -3 + 4t \rightarrow t = 1$

So $0 \leq t \leq 1$

Wednesday 2-oct-2024

⟹ The distance from apoint to aline in space = d

(المسافة بين S & L (بتكون عمودية

P : Point in L , $\vec{v}$: Parallel to L

$\vec{ps}$: Vector

$$d = |\vec{ps}| \sin\theta \frac{|\vec{v}|}{|\vec{v}|}$$

الزاوية بين ps & L

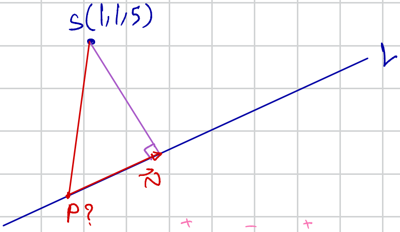
= الزاوية بين ps & $\vec{v}$

$$\therefore d = \frac{\vec{ps} \times \vec{v}}{|\vec{v}|}$$

example: Find The distance from S(1,1,5) to Line L , Parametric equation " x = 1+t , y = 3−t , z = 2t " −∞ < t < ∞

to find p ∂ t = 0 ⟹ x = 1 , y = 3 , z = 0 p(1,3,0)

$\vec{Ps} = \vec{S} - \vec{P} = 0i - 2j + 5k = \underline{\underline{-2j + 5k}}$ ✓



to find The Vector $\vec{v}$, depend on Parametric equation

$x = x_0 + t v_1$

$y = y_0 + t v_2$

$z = z_0 + t v_3$

So we Conclude That $\vec{v}$ is the Coefficient of t

$\Rightarrow \vec{v} = \hat{i} - \hat{j} + 2k$ ✓

$|\vec{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

$$\vec{ps} \times \vec{v} = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\vec{Ps} \times \vec{v} = \hat{i} + 5\hat{j} + 2\hat{k}$$

$$|\vec{Ps} \times \vec{v}| = \sqrt{1^2 + 5^2 + 2^2} = \sqrt{30}$$

$$\therefore d = \frac{\vec{Ps} \times \vec{v}}{|\vec{v}|} = \frac{\sqrt{30}}{\sqrt{6}} = \boxed{\sqrt{5}}$$

- an equation for a plane in space

$\vec{P_0P} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$

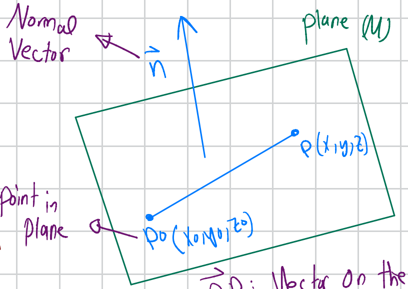
$\vec{n} = Ai + Bj + Ck$

↳ ⊥ Plane (M) , $\vec{n} \perp \vec{P_0P}$

$\vec{P_0P} \cdot \vec{n} = 0$ " Vector equation "

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ " Component equation "

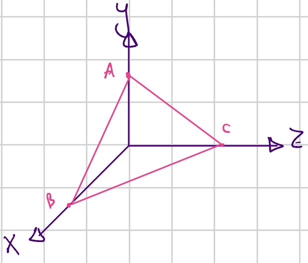
$$Ax + By + Cz = \underbrace{Ax_0 + By_0 + Cz_0}_{D}$$



$\vec{P_0P}$: Vector on the plane

$\therefore Ax + By + Cz = D$ —* " The plane equation

↳ The Component equation Simplified for the plane passing Through p0(x0,y0,z0) and the normal $\vec{n}$ = Ai + Bj + Ck example: find an equation for the plane passing through A(0,0,1), B(2,0,0), C(0,3,0)



$\vec{n} = \vec{AB} \times \vec{AC}$

$= (2i - k) \times (3j - k)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \overset{a}{\underset{=}{3}}\hat{i} + \overset{B}{\underset{=}{2}}\hat{j} + \overset{c}{\underset{=}{6}}k$$

$n \perp$ plane ✓, point on the plane (we can take any point) $\Longrightarrow A(\overset{x_0,\,y_0,\,z_0}{0,0,1})$

$\Rightarrow$ The equation of The plane:

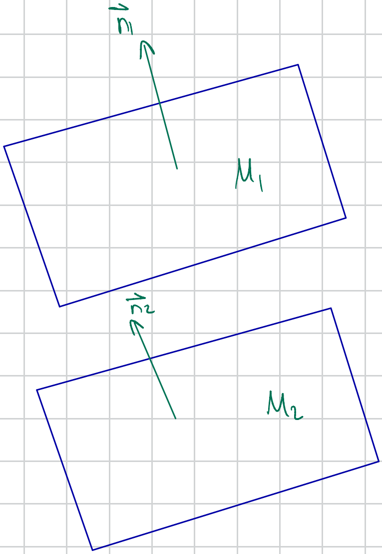
$3(x-0) + 2(y-0) + 6(z-1) = 0$

$3x + 2y + 6z - 6 = 0 \Longrightarrow 3x + 2y + 6z = 6$ —✱

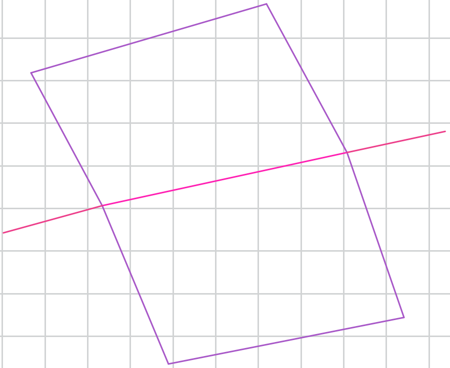
✱ line of intersection:

① Two planes are parallel IFF Their normal are parallel.

$\Rightarrow$ That is M_1 parallel to M_2 IFF $\vec{n}_1 = \underset{\underset{\text{Scalar.}}{\downarrow}}{k}\,\vec{n}_2$



② Two planes that are not parallel intersect in a line



Saturday 5-oct-2024

example: find a vector parallel to the line of intersection of the planes

Plane 1 : $3x - 6y - 2z = 15$, plane 2 : $2x + y - 2z = 5$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$\vec{n}_1 = 3i - 6j + 2k$ $\qquad$ $\vec{n}_2 = 2i + j - 2k$

$\vec{v} \parallel L = \vec{n}_1 \times \vec{n}_2$ "vector parallel to this line (line of intersection)"

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix} = \overset{v_1}{14i} + \overset{v_2}{2j} + \overset{v_3}{15k}$$

- find parametric equation for the line of intersection. $(14i + 2j + 15k)$

→ put $z = 0$ ⟹

$$\begin{aligned} & 3x - 6y = 15 \\ + \; & 2x + y = 5 \times 6 \\ \hline & 15x = 45 \Rightarrow x = 3 \Rightarrow y = -1 \end{aligned}$$

⟹ $p(\overset{x_0}{3}, \overset{y_0}{-1}, \overset{z_0}{0})$ on the line intersection

⟹ parametric equation of line : $x = 3 + 14t$, $y = -1 + 2t$, $z = 15t$

example: find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersect the plane $3x + 2y + 6z = 6$

plane ⟸ Line معادلة يعوض .

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + \cancel{6} + 6t = \cancel{6} \Rightarrow 8t = -8 \Rightarrow t = -1$$

$t = -1 \Rightarrow x = \frac{8}{3} - 2 \Rightarrow x = \frac{2}{3}$

$y = 2$, $z = 0$

∴ point of intersection $\left(\frac{2}{3}, 2, 0\right)$

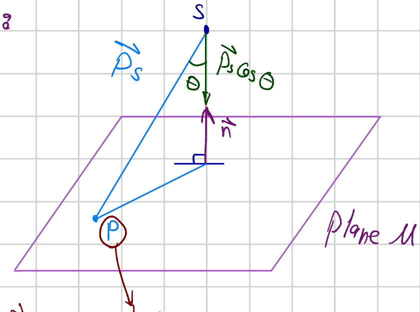
* The distance from a point to a plane :

M : $Ax + By + Cz = 0 = D$

$\vec{n} = Ai + Bj + Ck$

A, B, C → Components of normal vector

1) $\vec{n} = Ai + Bj + Ck$
2) $P = (x, y, z)$
3) $\vec{PS}$ "vector"



نقطة ع plane

قيمة مطلقة

$$d = \vec{PS} \cos\theta \cdot \frac{|\vec{n}|}{|\vec{n}|} \Rightarrow d = \frac{|\vec{PS} \cdot \vec{n}|}{|\vec{n}|}$$

example : Find the distance from $S(1,1,3)$ to the plane $3x+2y+6z=6$

$\vec{n} = 3i + 2j + 6k$

to find the point on the plane put $x=y=0$ on the plane equation , So $\underline{z=1}$

$P = \vec{k} \Rightarrow \vec{PS} = i + j + 2k$

$P=(0,0,1)$

$$\Rightarrow d = \frac{|\vec{PS}\cdot\vec{n}|}{|\vec{n}|} = \frac{|3+2+12|}{\sqrt{9+4+36}} \Rightarrow \therefore d = \frac{17}{7}$$

* Angle between two planes

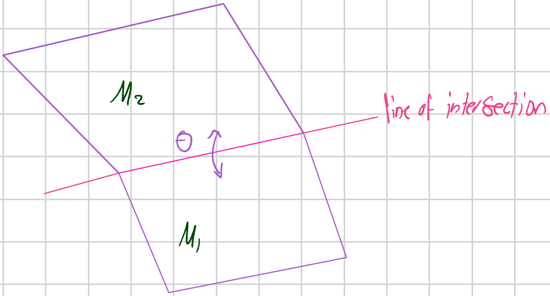
= The angle between their normal Vectors

$M_1 \Leftarrow n_1 = A_1 i + B_1 j + C_1 k$

$M_2 \Leftarrow n_2 = A_2 i + B_2 j + C_2 k$

$n_1 \cdot n_2 = |n_1| \cdot |n_2| \cos\theta$

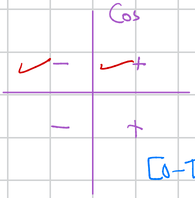
$$\Rightarrow \cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|n_1|\cdot|n_2|}$$



example: find The angle between $M_1 = 3x - 6y - 2z = 15$, $M_2 = 2x + y - 2z = 5$

$\vec{n_1} = 3i - 6j - 2k$, $\vec{n_2} = 2i + j - 2k$

$$\theta = \cos^{-1}\left(\frac{6-6+4}{\sqrt{49}\sqrt{9}}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{4}{21}\right)$$

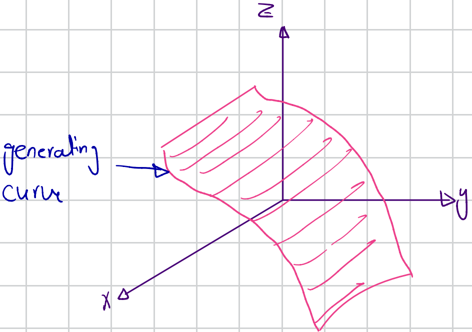


بال $\cos^{-1}$ مختار
بس الربع الاول
والثاني لأنه $\cos^{-1}$
معرفة بس من $[0-\pi]$

Saturday 5-Oct-2024

12.6: cylinders and quadric surfaces

Cylinders: is a surface generated by moving a straight line along a given planer curve while holding the line parallel to given fixed line.



Parallel to x-axis

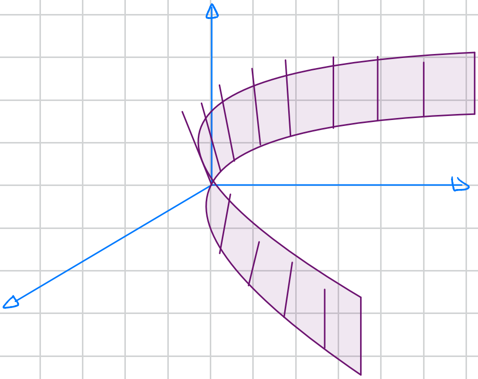
Example: Find an equation for the cylinder made by the lines parallel to the z axis that pass through the parallel $y = x^2$, $z = 0$ → generating curve.

↳ fixed line.

← تحرك line على Curve ينتج الشكل

(x, x^2, z)

↳ The cylinder $y = x^2$



- اي Surface بنتج معنا Cylinder.
- معادلة الكرة (sphere) ← $x^2+y^2+z^2=a^2$

Monday

بسم الله الرحمن الرحيم

14-Oct-2024

Chapter 13: Vector-valued functions and motion in space

13.1: curves in space and their tangents

→ The parameterization of this curve:

$x = F(t)$, $y = g(t)$, $z = h(t)$

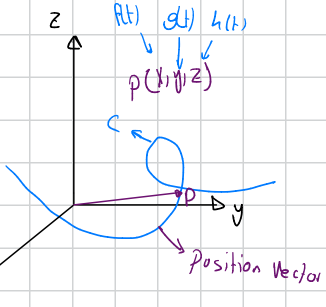
* when a particle p moves through the space during the time (t), the the coordinate of this particle defined as $x = F(t), y = g(t), z = h(t)$

" parametric equations of c "

* The set C of all points:

$(x, y, z) = (f(t), g(t), h(t))$, $t \in \boxed{I}$ → interval.

is called a "space curve"



* The position vector from the origin to any point on the vector $\vec{r} = \vec{op}$

$\vec{r}(t) = F(t)i + g(t)j + h(t)k$

position vector ← يعطيك موقع الجسم

- a curve in space can be also represented in vector form $\vec{r} = \vec{op}$

$= \langle x, y, z \rangle$

$= \langle f(t), g(t), h(t) \rangle$

Def: Vector function is a function where domain is a set of real numbers and range is a set of vectors

$\vec{r}(t) = f(t)i + g(t)j + h(t)k$ $\quad t \to \vec{r}(t)$

- Real valued functions are called scaler functions
- Components of $\vec{r}$ $\{f(t), g(t), h(t)\}$ are scaler functions
- The Domain of $\vec{r}$ is the common domian of it's components

$D(\vec{r}) = D(f) \cap D(g) \cap D(h)$

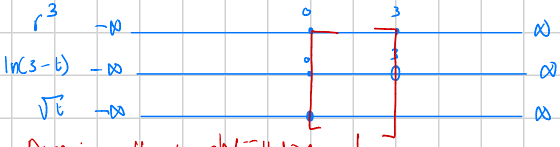
example: $\vec{r}(t) = t^3 i + \ln(3-t) j + \sqrt{t} k$, find a) $\vec{r}(2)$ b) $D(\vec{r}(t))$

a) $\vec{r}(2) = 2^3 i + \underset{0}{\cancel{\ln(3-2) j}} + \sqrt{2} k$

$= 8i + \sqrt{2} k$

b) $D(\vec{r}(t))$

- $D(r^3) \Rightarrow \mathbb{R}^3$
- $D(\ln(3-t)) \Rightarrow 3 - t > 0 \Rightarrow t < 3$
- $D(\sqrt{t}) \Rightarrow t \geq 0$



→ Domains حل التقاطع بين ال ←

$\therefore D(\vec{r}(t)) = [0, 3)$

example: graph the Vector function

$\vec{r}(t) = (\cos t) i + (\sin t) j + t k$

$x = \cos t \; , \; y = \sin t \; , \; z = t$

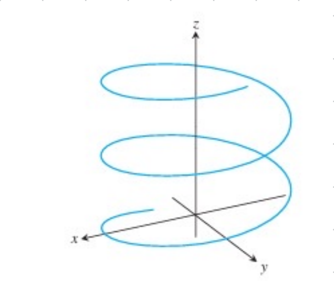
$\rightarrow \; x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$

$\Rightarrow \; x^2 + y^2 = 1$

$\Rightarrow \; z = t$

if $t = 0$, $\vec{r}(0) = 1i + 0j + 0k$
$= i$

if $t = \frac{\pi}{2}$, $\vec{r}(\frac{\pi}{2}) = 0i + 1j + \frac{\pi}{2} k$
$= j + \frac{\pi}{2} k$



r(t) = (cos t)i + (sin t)j + tk

---

* limits and Continuity:

$\vec{r}(t) = x(t) i + y(t) j + z(t) k$

$$\lim_{t \to t_0} \vec{r}(t) = \left(\lim_{t \to t_0} x(t) \right) i + \left(\lim_{t \to t_0} y(t) \right) j + \left(\lim_{t \to t_0} z(t) \right) k$$

- Def : Continuity : A Vector function $\vec{r}(t)$ is Contininous at $t = t_0$ if $\lim_{t \to t_0} (\vec{r}(t)) = \vec{r}(t_0)$

example: if $\vec{r}(t) = \overset{x}{\underline{\left(\frac{t^2-1}{t-1} \right)}} i + \overset{y}{\underline{\frac{\sin t}{t}}} j + \overset{z}{(\underline{\ln t})} k$ find,

1) $D(\vec{r}(t))$ 2) $\lim_{t \to 1} \vec{r}(t)$ 3) where $\vec{r}(t)$ is Continious ?

1) $D(x(t)) \Rightarrow t \neq 1$

$D(y(t)) \Rightarrow t \neq 0$

$D(z(t)) \Rightarrow t > 0$

$\therefore D(\vec{r}(t)) = (0, \infty) / \{1\}$ or $(0,1) \cup (1, \infty)$

$\ln 1 = 0$

2) $\lim_{t \to 1} \vec{r}(t) = \lim_{t \to 1} \left(\frac{t^2-1}{t-1} \right) i + \lim_{t \to 1} \left(\frac{\sin t}{t} \right) j + \lim_{t \to 1} (\underset{0}{\cancel{\ln t}}) k$

using Loptial rule

$= \frac{2t}{1} = 2t i$

$= 2t i + \sin 1 j$

3) $r(t)$ is cont. on it's Domain
$(0,1) \cup (1, \infty)$

Example: $\vec{r}(t) = (\cos t)i + (\sin t)j + [t]k$, is Cont.?

- Sin & Cos ⇒ Cont $\mathbb{R}^3$
- $[t]$, step = 1

$$[t] = \begin{cases} -2, & -2 \leq t < -1 \\ -1, & -1 \leq t < 0 \\ 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \end{cases}$$

$[t]$ greatest integer function اكبر عدد صحيح

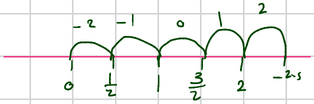
1) find the step $= \frac{1}{\text{معامل } t}$

2) $[t] = 0$ عشان اشوف وين ابلش

ex: $[2t-2] = 0 \Rightarrow t=1$

step = $\frac{1}{2}$

ـه بيلش من 1



$\Rightarrow D(\vec{r}(t)) = \mathbb{R}$

$\vec{r}(t)$ is Cont. on

$(-\infty, \infty) / \{\mp 1, \mp 2, \dots\}$

or $(-\infty, \infty) / \mathbb{Z}$ "الاعداد الصحيحة" "include zero"

---

* Derivatives and Motion:

Def: the Vector function

$\vec{r}(t) = f(t)i + g(t)j + h(t)k$

has a derivative at t if f,g,h have derivatives at t

$\vec{r}'(t) = f'(t)i + g'(t)j + h'(t)k$

Example: $\vec{r}(t) = (\ln t)i + \frac{t+1}{t+2}j + t\ln t\,k$, find $\frac{d\vec{r}}{dt}$ at $t=1$

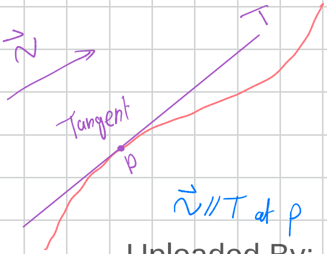
$$\frac{d\vec{r}}{dt} = \frac{1}{t}i + \frac{(t+2)(1)-(t+1)(1)}{(t+2)^2} + t\left(\frac{1}{t}\right) + \ln t\,(1)$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=1} = \frac{1}{1} + \frac{(1+2)-(1+1)}{(1+2)^2} + 1\left(\frac{1}{1}\right) + \cancel{\ln 1}\ 0$$

$$= 1i + \frac{3-2}{9}j + 1k \Longrightarrow \left.\frac{dr}{dt}\right|_{t=1} = i + \frac{1}{9}j + k$$

Remark: The tangent line to the curve at a point $p(f(t_0), g(t_0), h(t_0))$ is defined to be the line through this point

P & T parallel to $\vec{v} = \left.\frac{d\vec{r}}{dt}\right|_{t=t_0}$



$\vec{v} \parallel T$ at p

example: in the previous example, find the parametric equation of the tangent line at $t=1$

1) $\vec{r}(1) = (\underset{0}{\cancel{\ln 1}})i + \frac{1+1}{1+2}j + \underset{0}{\cancel{1\ln 1}}\,k \Rightarrow \therefore \vec{r}(1) = \frac{2}{3}j = (x_0, y_0, z_0)$

2) $r'(1) = i + \frac{1}{9}j + k$ which equal $\langle v_1, v_2, v_3 \rangle = \vec{v}$

3) parametric equations

$x = x_0 + t v_1 \Rightarrow x = t$

$y = y_0 + t v_2 \Rightarrow y = \frac{2}{3} + \frac{1}{9}t$

$z = z_0 + t v_3 \Rightarrow z = t$

Remark: ① A vector function $\vec{r}(t)$ is differentiable if it is differentiable at every point of it's domian

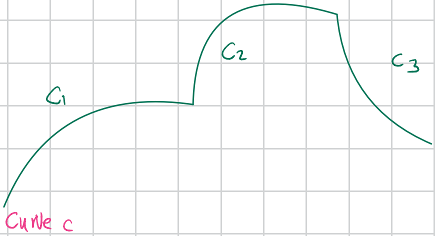
② The curved traced by $\vec{r}(t)$ is Smooth if $\frac{d\vec{r}}{dt}$ is cont. and $\frac{d\vec{r}}{dt} \neq 0$

③ The curve is called piece – wise smooth if it's made up of a finite number of Smooth Curves

example

c_1 & c_2 & c_3 ⇒ are smooth

But c ⇒ piece-wise Smooth



Definition: $\vec{r}(t)$ position vector of particle moving along smooth curve in space then:

$\vec{v}(t) = \frac{d\vec{r}}{dt}$ ⇒ Velocity Vector , $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2 r}{dt^2}$ ⇒ acceleration Vector.

Speed $= |\vec{v}|$, direction of motion $= \frac{\vec{v}}{|\vec{v}|}$ } Unit Vector

Example: $\vec{r}(t) = e^{-t}\, i + 2(\cos 3t)\, j + (2\sin 3t)\, k$ "position Vector"

Find:

① Velocity : $\vec{v}(t) = \frac{d\vec{r}}{dt}$

$$= -e^{-t}\, i + (-6\sin 3t)\, j + (6\cos 3t)\, k$$

at $t=0$ ⇒ $\underline{\underline{-i + 6k}}$

② speed $= |\vec{v}| = \sqrt{(-e^{-t})^2 + \underbrace{(-6\sin 3t)^2 + (6\cos 3t)^2}_{\sin^2 3t + \cos^2 3t = 1}} = \sqrt{e^{-2t} + 36}$

at $t=0$ ⇒ $\underline{\underline{\sqrt{37}}}$

③ acceleration $\vec{a}(t) = \frac{d\vec{v}}{dt}$

$$= e^{-t}\, i + (-18\cos 3t)\, j + (-18\sin 3t)\, k$$

at $t=0$ ⇒ $\underline{\underline{i - 18j}}$

④ direction of motion at $t=0$ $\left.\frac{\vec{v}}{|\vec{v}|}\right|_{t=0}$

$$\frac{-i + 6k}{\sqrt{37}} = \frac{-1}{\sqrt{37}}\, i + \frac{6}{\sqrt{37}}\, k$$

Example: $\vec{r}(t) = 4\cos(\frac{t}{2})\, i + 4\sin(\frac{t}{2})\, j$, Find a) $\vec{v}(\pi)$ b) $\vec{a}(\pi)$

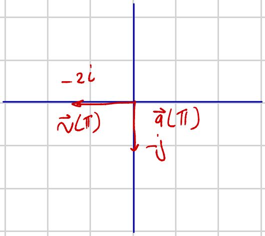
a) $\vec{v}(t) = -2\sin(\frac{t}{2})\, i + 2\cos(\frac{t}{2})\, j$

$\vec{v}(\pi) = -2i$

Then sketch them as Vectors

b) $\vec{a}(t) = -\cos(\frac{t}{2})\, i - \sin(\frac{t}{2})\, j$

$\vec{a}(\pi) = -j$



- Differentiation Rules:

① $\frac{d}{dt}\vec{c} = \vec{0}$ ؛ ن $\vec{c}$: Constant Vector , $\vec{c} = c_1 i + c_2 j + c_3 k$

② $\frac{d}{dt}(c\,\vec{u}(t)) = c\,\vec{u}'(t)$

③ $\frac{d}{dt}(\underset{\text{Scaler function}}{F(t)}\ \overset{\text{Vector function}}{\vec{u}(t)}) = F'(t)\,\vec{u}(t) + \vec{u}'(t)\,\vec{F}(t)$

④ $\frac{d}{dt}(\vec{v}(t) \pm \vec{u}(t)) = \frac{d}{dt}\vec{v}(t) \mp \frac{d}{dt}\vec{u}(t)$

⑤ $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \left.\vec{u}(t)\,\vec{v}'(t) + \vec{v}(t)\,\vec{u}'(t)\right\}$ Scaler function

⑥ $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}(t) \times \vec{v}'(t) + \vec{v}(t) \times \vec{u}'(t)$

⑦ $\frac{d}{dt}(\vec{u}(F(t))) = F'(t)\,\vec{u}'(t)$

13.2: integrals and vector function

- Integrals of Vector functions

- indefinite integral
  $\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$ → ثابت التكامل

- definite integral
  $\int_a^b \vec{r}(t)\,dt = \left.\vec{R}(t)\right|_a^b = \vec{R}(b) - \vec{R}(a)$

example: $\frac{dr}{dt} = (180t)i + (180t - 16t^2)j$, initial condition $\vec{r}(0) = 100j$, find $\vec{r}$
بكامل

$$\vec{r} = \int \left(\frac{dr}{dt}\right) dt = 90t^2 i + (90t^2 - \frac{16}{3}t^3)j + C$$

$$\vec{r}(0) = 0i + (0 - 0)j + C = 100 \Rightarrow \therefore \underline{\underline{C = 100}}$$

$$\therefore \vec{r} = 90t^2 i + (90t^2 - \frac{16}{3}t^3 + 100)j$$

Wednesday 30-Oct-2024

13.3: Arc length in space

$\vec{r}(t) = x(t)\,i + y(t)\,j + z(t)\,k$ $\quad a \leq t \leq b$

$\Rightarrow$ Arc length of $\vec{r}$ from $t=a$ to $t=b$ is

$$L = \int_a^b \boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}\, dt$$

$\Rightarrow$ The Same of $|v|$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,i + \frac{dy}{dt}\,j + \frac{dz}{dt}\,k$$

$$\boxed{L = \int_a^b |v|\, dt}$$

example: find the length of the curve $\vec{r}(t) = (t\cos t)i + (t\sin t)j + \frac{2\sqrt{2}}{3}\, t^{3/2}\, k$ $\quad 0 \leq t \leq \pi$

- $\frac{dx}{dt} = t(-\sin t) + \cos t \Rightarrow \left(\frac{dx}{dy}\right)^2 = t^2\sin^2 t - 2(t\sin t)(\cos t) + \cos^2 t$

- $\frac{dy}{dt} = t(\cos t) + \sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = t^2\cos^2 t + 2(t\cos t)(\sin t) + \sin^2 t$

- $\frac{dz}{dt} = \frac{\cancel{2}\sqrt{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}}\, t^{1/2} = \sqrt{2}\, t^{1/2} \Rightarrow \left(\frac{dz}{dt}\right)^2 = 2t$

$\sin^2 t + \cos^2 t = 1$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \Longrightarrow t^2\sin^2 t - \cancel{2t\sin t\cos t} + \cos^2 t + t^2\cos^2 t + \cancel{2t\cos t\sin t} + \sin^2 t + 2t$$

$$t^2(\sin^2 t + \cos^2 t) \Rightarrow t^2(1) = t^2$$

$$= t^2 + 2t + 1 = (t+1)^2$$

$|t+1| = \begin{cases} t+1, & t \geq -1 \\ -(t+1), & t < -1 \end{cases}$

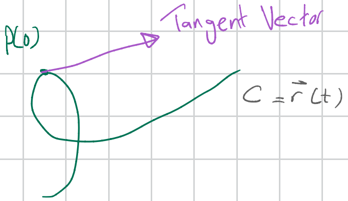
$$L = \int_a^b \sqrt{\boxed{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}\, dt$$

$\Rightarrow$ equal $(t+1)^2$

$$= \int_0^\pi \sqrt{(t+1)^{\cancel{2}}}\, dt \Rightarrow \int_0^\pi |t+1|\, dt = \int_0^\pi t+1\, dt \Rightarrow \Big|_0^\pi \frac{t^2}{2} + t$$

$$= \boxed{\frac{\pi^2}{2} + \frac{\pi}{2}}$$

* Unit tangent Vector



$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$\vec{T}$ is called a unit tangent vector to the smooth curve c , $\frac{dr}{dt} \neq 0$ & cont

example: find a unit tangent vector

$\vec{r}(t) = 3\cos t\, i + 3\sin t\, j + t^2 k$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -3\sin t\, i + 3\cos t\, j + 2t\, k$$

$$|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}\ dt$$

Knowing that $\sin^2 t + \cos^2 t = 1$

$$= \sqrt{9 + 4t^2}$$

$$\therefore \vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{(-3\sin t)i + (3\cos t)j + 2t\,k}{\sqrt{9+4t}}$$

* Arc length parameter with Base point p(t₀)

$$S(t) = \int_{t_0}^{t} \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\ d\tau$$

$$S(t) = \int_{t_0}^{t} \underbrace{|v(\tau)|}_{\text{velocity}}\, d\tau$$

, S(t) is a Scaler function

example: $\vec{r}(t) = 4\cos t\, i + 4\sin t\, j + 3t\, k$

بشتق $\vec{r}(t)$ وبربع كل حد لحاله
وبحطه تحت الجذر

a) find S(t) with base point p(0)

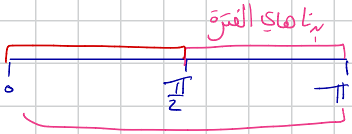
$$S(t) = \int_0^t \sqrt{\underbrace{16\sin^2 t + 16\cos^2 t}_{16(1)} + 9}\ d\tau$$

$$= \int_0^t \sqrt{25}\, d\tau = \int_0^t 5\, d\tau = \Big|_0^t 5\tau \Longrightarrow \boxed{5t = S(t)}$$

b) length : $\frac{\pi}{2} \leq t \leq \pi$

$$L = \int_0^{\pi} = \int_{\frac{\pi}{2}}^{0} + \int_0^{\pi}$$

$$\int_{\frac{\pi}{2}}^{\pi} = \int_0^{\pi} - \int_0^{\frac{\pi}{2}}$$



$$\int_{\frac{\pi}{2}}^{\pi} = \int_0^{\pi} - \int_0^{\frac{\pi}{2}}$$

$$\boxed{\begin{aligned} L &= S(\pi) - S(\tfrac{\pi}{2}) \\ L &= 5\pi - \tfrac{5}{2}\pi \end{aligned}}$$

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds}$$

$$S = \int_{t_0}^{t} |v| \, dt \Longrightarrow \frac{ds}{dt} = |v| \Longrightarrow \frac{dt}{ds} = \frac{1}{|v|}$$

مشتقة التكامل
يساوي ما داخل التكامل

$$\frac{d\vec{r}}{ds} = \vec{v} \cdot \frac{1}{|v|}$$

$$\boxed{\therefore \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|v|} = T}$$

Unit tangent Vector

Example: if $\vec{r}(t) = \cos t\, i + \sin t\, j + tk$, find $\vec{T}$ using $\frac{d\vec{r}}{ds}$ taking $t_0 = 0$

$$S(t) = \int_0^t \sqrt{\underbrace{(\sin t)^2 + (\cos t)^2}_{=1} + 1} \, dt \Longrightarrow S(t) = \int_0^t \sqrt{2}\, dt \Longrightarrow S(t) = \sqrt{2}\, t$$

- $S = \sqrt{2}\, t \Longrightarrow t = \frac{S}{\sqrt{2}} \Longrightarrow \boxed{\frac{dt}{ds} = \frac{1}{\sqrt{2}}}$

- $\boxed{\frac{d\vec{r}}{dt} = -\sin t\, i + \cos t\, j + 1k}$

$$\Rightarrow \vec{T} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = -\sin t\, i + \cos t\, j + k \cdot \frac{1}{\sqrt{2}}$$

$$\boxed{\therefore \vec{T} = \frac{-\sin t\, i}{\sqrt{2}} + \frac{\cos t\, j}{\sqrt{2}} + \frac{k}{\sqrt{2}}}$$ #

example: find $\underset{L}{\underline{\text{Arc length}}}$ of the curve :

$$\vec{r}(t) = (\underbrace{\sqrt{2}\,t}_{x})\,i + (\underbrace{\sqrt{2}\,t}_{y})\,j + (\underbrace{1-t^2}_{z})\,k \text{ , from } \overset{x\,y\,z}{(0,0,1)} \text{ to } \overset{x\quad y\quad z}{(\sqrt{2},\sqrt{2},0)}$$

Values of t:

(x) $(0,0,1) \Rightarrow \sqrt{2}t = 0 \Rightarrow \boxed{t=0}$

(x) $(\sqrt{2},\sqrt{2},0) \Rightarrow \sqrt{2}t = \sqrt{2} \Rightarrow \boxed{t=1}$

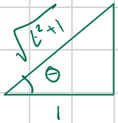
$$L = \int_a^b |v(t)|\,dt$$ a & b Values of t

$$L = \int_0^1 \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (-2t)^2}\;dt$$

$$= \int_0^1 \sqrt{2+2+4t^2}\;dt \Rightarrow \int_0^1 \sqrt{4+4t^2}\;dt \Rightarrow \int_0^1 2\sqrt{1+t^2}$$ → trigonometric Substiation

Let $t = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dt = \sec^2\theta\,d\theta$ $\quad$ 1st & 4th Qua



$$\Rightarrow \int \sqrt{1+t^2}\;dt = \int \sqrt{1+\tan^2\theta}\;(\sec^2\theta)\,d\theta$$

$$= \int \sqrt{\sec^2\theta}\;\sec^2\theta\,d\theta$$

$$= \int |\sec\theta|\,\sec^2\theta = \int \sec^3\theta$$ integral by parts

Let $u = \sec\theta$ $\qquad dv = \sec^2\theta\,dt$

$du = \sec\theta\tan\theta\,d\theta$ $\qquad v = \tan\theta$

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta$$

$$\int \sec^3\theta = \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\,d\theta$$

$$\int \sec^3\theta = \sec\theta\tan\theta - \cancel{\int \sec^3\theta} + \ln|\sec\theta + \tan\theta|$$

$$+\int \sec^3\theta \qquad\qquad +\cancel{\int \sec^3\theta}$$

$$\int \sec^3\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$t = \tan\theta$

if $t = 0 \Rightarrow \theta = 0$

if $t = 1 \Rightarrow \theta = \frac{\pi}{4}$

ن $L = ②\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta \Rightarrow \Big[\cancel{2}(\cancel{\tfrac{1}{2}}\sec\theta\tan\theta) + \cancel{\tfrac{1}{2}}\ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{4}}$

من قبل

ن $L = \sqrt{2} + \ln(\sqrt{2}+1) - (0 + \underset{0}{\cancel{\ln(1)}})$

$$\boxed{\Rightarrow L = \sqrt{2} + \ln(\sqrt{2}+1)}$$ #

Monday

بسم الله الرحمن الرحيم

4-Nov-2024

Chapter 14: Partial derivatives

14.1: functions of several variables

$z = f(x, y)$

dependent ← z ; x, y → independent

ممكن نشتق z يا بالنسبة لـ x او بالنسبة لـ y وممكن بالنسبة لـ x&y

Def: Suppose D is a set of n-tuples real numbers $(x_1, x_2, \dots, x_n)$ → scaler function

a real Valued function f on D is a Rule that assigns a single Value of Real numbers $w = f(x_1, x_2, \dots, x_n)$

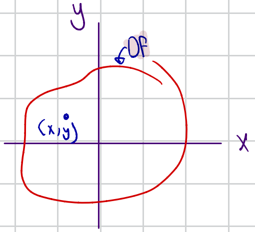
- D is the Domain of f
- The Value of w is the Range of f

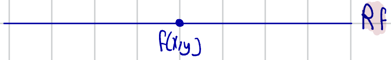
Note

* Domain هو مجموعة الأزواج المرتبة وبحدده في المستوى الديكارتي (y Values)

* Range هو عدد حقيقي وبحدده على خط الأعداد (z Values)

example: if f is a function of two indpendent Variables $z = f(x, y)$ Then we picture the Domain (D) as origin in xy-plane





example: $w = f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, find $f(3, 0, 4)$

$f(3, 0, 4) = \sqrt{3^2 + 0^2 + 4^2} = \sqrt{25} = \underline{\underline{5}}$ → Real Value function

example: find the Domain D and sketch it and find R

$\boxed{1}$ $z = f(x, y) = \sqrt{y - x^2}$

$y - x^2 \geq 0 \Rightarrow y \geq x^2$

$D_f = \{(x, y) : y \geq x^2\}$

$R_f = [0, \infty)$

un bounded and closed

$\boxed{2}$ $z = f(x,y) = \frac{1}{xy}$

$x=0 \rightarrow y\,axis,\ y=0 \rightarrow x\,axis$

$Df = \{(x,y) : xy \neq 3 \Rightarrow Df = \{(x,y) : x \neq 0 \ \& \ y \neq 0\}$

$Rf = \mathbb{R}/0$

كل أشي الا Xaxis
& Yaxis

Unbounded
open

$\boxed{3}$ $f(x,y) = \sqrt{9-x^2-y^2}$

to find D

$9-x^2-y^2 \geq 0$

$9 \geq x^2+y^2 \Rightarrow x^2+y^2 \leq 9$

$Df = \{(x,y) : x^2+y^2 \leq 9\}$

to find R

$0 \leq x^2+y^2 \leq 9$

$0 \geq -x^2-y^2 \geq -9$

$9 \geq 9-x^2-y^2 \geq 0$

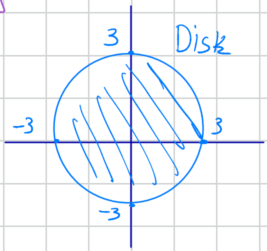
$\sqrt{9} \geq \sqrt{9-x^2-y^2} \geq \sqrt{0}$

$3 \geq \underbrace{\sqrt{9-x^2-y^2}}_{z} \geq 0$

$Rf = [0,3]$

لما اكون بدي اوجد Range باخذ الـ D
وبحوله او بصل للـ Function الاصلي
عشان الاقي الـ Range

bounded
& closed



$\boxed{4}$ $f(x,y) = 4\sin^{-1}(y-2x)$

$\sin^{-1}x \Rightarrow Df = -1 \leq x \leq 1$

$Rf = -\frac{\pi}{2} \leq \sin^{-1} \leq \frac{\pi}{2}$

$Df = \{(x,y) : 2x-1 \leq y \leq 2x+1\}$

$Rf = [-2\pi, 2\pi]$

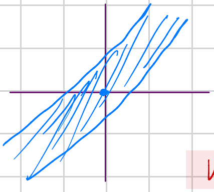
$-1 \leq y-2x \leq 1$

$2x-1 \leq y \leq 2x+1$

$-\frac{\pi}{2} \leq \sin^{-1}(y-2x) \leq \frac{\pi}{2}$

$-2\pi \leq 4\sin^{-1}(y-2x) \leq 2\pi$

$\sin^{-1}$ لا تختلف مهما
اختلفت ما بداخلها يعني
D&R for x = D&R for y-2x



unbounded & closed

$\boxed{5}$ $z = \sin(xy)$

Domain : $\sin(xy)$ is defined for all real numbers x & y oR (The entire plane)

Range : $-1 \leq \sin t \leq 1$

$-1 \leq \sin(\underline{xy}) \leq 1 \Rightarrow$ $Df = \mathbb{R}$

بغض النظر عما داخل ال sin

$R = [-1,1]$

$\boxed{6}$ $w = \sqrt{x^2+y^2+z^2}$

Domain $x^2+y^2+z^2 \geq 0 \rightarrow \{(x,y,z) : x^2+y^2+z^2 \geq 0\}$

Range $[0, \infty)$

$\boxed{7}$ $w = \dfrac{1}{x^2+y^2+z^2}$ $\qquad x^2+y^2+z^2 \neq 0$

Domain $(-\infty, 0) \cup (0, \infty)$ or $\mathbb{R} / \{0\}$

Range $(0, \infty)$

↳ ما بطلع -ve لانه تربيع

$\boxed{8}$ $w = \underset{\mathbb{R}}{x}\,\underset{\mathbb{R}}{y} \ln z \rightarrow z > 0$

Domain $\{(x,y,z) : \in \mathbb{R} / z > 0\}$

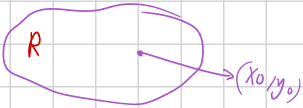
Range $xy \ln z$; $xy\ (-\infty, \infty)$; $\ln z\ (-\infty, \infty)$

is $R = (-\infty, \infty)$

* function of two Variables

Def: • A point (x,y) in a region R in the xy-plane is an interior point of R if it is the Center of a Disk that lies entirely in R

• A point (x_0, y_0) is a Boundary point of R if every disk Centered at (x_0, y_0) Contains points that lies outside R and points as well that lie in R
يكون



- The interior of R is the set of all interior points of R
- The boundary of R is the set of all boundary points of R
- The Region R is open if it Consists entirely of interior point
- The Region R is Closed if it Contains all its boundary points
- A region in the plane is bounded if it lies inside a disk of fixed radius
- A region is unbounded if it is not bounded

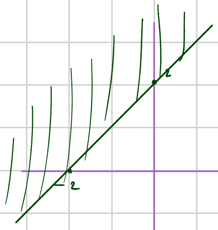
→ Bounded → line segments, Triangles, rectangles, circles, disk

→ unBounded → lines, Coordinate axes, quadrants, half-planes, the plane.

example: find and sketch the domain of :

1) $f(x,y) = \sqrt{y-x-2}$; $y-x-2 \geq 0 \Rightarrow y \geq x+2$

$\therefore$ Domain : $\{(x,y) : y \geq x+2\}$



$y=1 \rightarrow x \leq -1$

$y=2 \rightarrow x \leq 0$

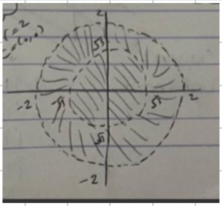
$y=3 \rightarrow x \leq 1$

2) $f(x,y) = \dfrac{1}{\ln(4-x^2-y^2)}$

$4-x^2-y^2 > 0$ and $\ln(4-x^2-y^2) \neq 0$

$\underbrace{x^2+y^2 < 4}_{r=2,\ \text{center}=(0,0)}$ $\quad 4-x^2-y^2 \neq 0$

$\underbrace{x^2+y^2 \neq 3}_{r=\sqrt{3},\ c=(0,0)}$



$\therefore$ Domain $\{(x,y) : x^2+y^2 \neq 3 \ \& \ x^2+y^2 < 4\}$

⟸ في كمان مثال + في امثلة عى open & close في العرض تاريخ 16.Nov ⟹

* Graphs , level curve, contours of function of two independent variables

Def: 1 The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$ is called a level curve

2 The set of points $(x,y,f(x,y))$ in space is called the graph of f
$\quad$ z

↳ surface of f ($z=f(x,y)$)

example: find and sketch the level curves of (Contour curve or Maps)

1) $f(x,y) = 100-x^2-y^2$ at $C = 0, 75$

$z = 100-x^2-y^2$

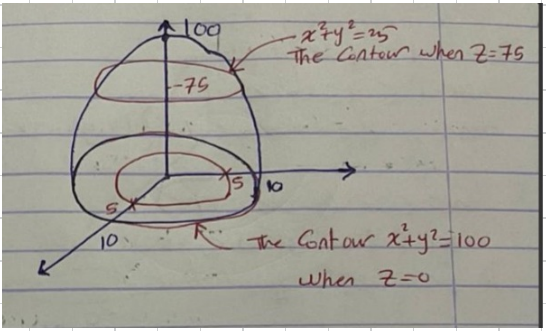
$x^2+y^2 = 100-z$ "Paraboloid"

$C=0 \Rightarrow 100-x^2-y^2 = 0$

$x^2+y^2 = 100$

$C=75 \Rightarrow 100-x^2-y^2 = 75$

$x^2+y^2 = 25$

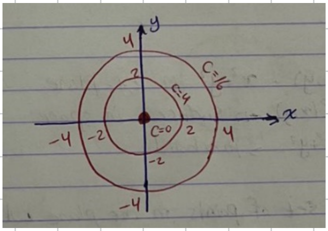


2) $f(x,y) = x^2+y^2$, $C = 0, 4, 16$

$C=0 \Rightarrow x^2+y^2=0$

$C=4 \Rightarrow x^2+y^2=4$

$C=16 \Rightarrow x^2+y^2=16$



example: Describe the level Curve of $z = f(x,y) = 100-x^2-y^2$

level curve $f(x,y) = c$

$100-x^2-y^2 = C$

$100-C = x^2+y^2$

- $C=100 \Rightarrow x^2+y^2=0 \Rightarrow (x,y)=(0,0) \Rightarrow$ level curve is origin point

-ve sign

- $C>100 \Rightarrow \underline{x^2+y^2<0}$ "No level Curve"

$100-C>0$

- $C<100 \Rightarrow x^2+y^2>0$ → level curve a circle with $C(0,0)$ and $r=\sqrt{100-C}$

$C-100<0$

Def: The set of points in space where a function of 3 variables has a Constant Value $f(x,y,z)=C$ is Called a level surface.

example: Describe level surface

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$\Rightarrow f(x,y,z) = c \Rightarrow \sqrt{x^2+y^2+z^2} = c \Rightarrow \boxed{x^2+y^2+z^2 = c^2}$

- $C>0$ (sphere)
  Center $(0,0,0)$
  Radius $(\sqrt{c^2}) = \underline{C>0}$

- $C=0$
  $x^2+y^2+z^2=0$
  point $(0,0,0)$

- $C<0$
  No Surface

16.Nov ⇐ بروفي كمان مثالين بالمعرض بتاريخ

Thursday　　　　بسم الله الرحمن الرحيم　　　　7-Nov-2024

14.2: limits and continuity in higher dimension

$$\lim_{x,y\to(x_0,y_0)} f(x,y) = ؟؟$$

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$$

→ Constant "exists"

→ Variable "paths"

example: 1) $\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \frac{0-0+3}{0+0-1} \Rightarrow \underline{\underline{-3}}$ ✓ exists

2) $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{9+16} = \underline{\underline{5}}$ ✓ exists

3) $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{0}{0}$ ← التعويض المباشر

"بضرب بمرافق المقام"

اذا بالتعويض المباشر طلع $\frac{0}{0}$ ⇐ مش شرط يطلع exists

لما يطلع الناتج بالتعويض المباشر $\frac{0}{0}$ بدور عن طريقة

لحل limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(x-y)} \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}-\sqrt{y})}{\cancel{(x-y)}}$$

$$\therefore \lim_{(x,y)\to(0,0)} x(\sqrt{x}-\sqrt{y}) \quad (\text{بعوّض}) \Longrightarrow$$ The answer is $\underline{\underline{\text{Zero}}}$ → constant ∴ exists

4) $\lim_{(x,y)\to(2,-4)} \frac{4+y}{x^2y-xy+4x^2-4x}$ التعويض المباشر $\frac{0}{0}$

$$\lim_{x,y\to(2,-4)} \frac{4+y}{xy(x-1)+4x(x-1)} \Longrightarrow \lim_{(x,y)\to(2,-4)} \frac{\cancel{4+y}}{(x-1)x\cancel{(y+4)}}$$

$$\therefore \lim_{(x,y)\to(2,-4)} \frac{1}{x^2-x} = \frac{1}{4-2} \Longrightarrow \therefore \lim_{(x,y)\to(2,-4)} = \frac{1}{2}$$ Constant ∴ exists

5) $\lim_{(x,y)\to(0,0)} y^2 \sin\left(\frac{1}{x}\right)$

$0 \sin(\mp\infty)$ بعد التعويض بطلع

$\sin x$ تتراوح بين $\underline{\underline{-1 \,\&\, 1}}$ فما بعرف ما

$\sin \mp\infty$ قديش

using Sandwich theorm

$$-1 \leqslant \sin\frac{1}{x} \leqslant 1 \quad , x \neq 0$$

$$-y^2 \leqslant y^2 \sin\frac{1}{x} \leqslant y^2 \Longrightarrow -y^2 \leqslant f(x,y) \leqslant y^2 \checkmark$$

① حققنا انو ال function حصرناه بين
نقطتين

$$\Rightarrow \lim_{(x,y)\to(0,0)} -y^2 = \lim_{(x,y)\to(0,0)} y^2 = 0 \checkmark$$

② نتأكد انو lim للطرفين $(-y^2, y^2)$ متساويين

$\therefore$ By Sandwich theorm $\lim_{(x,y)\to(0,0)} y^2 \sin\frac{1}{x} = 0$ $\therefore$ exists $\therefore$

6) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ $\quad \frac{0}{0}$ بالتعويض المباشر

بقدر استخدم الفرض لحل هذا ال Lim

Let $u = x^2+y^2$

$(x,y)\to(0,0) \quad \therefore \underline{\underline{u \to 0}}$

$\lim_{u\to 0} \frac{\sin u}{u}$ used lopital Rule

$\lim_{u\to 0} \frac{\cos u}{1} = \cos(0) = \underline{\underline{1}}$ $\therefore$ exists

$\Leftarrow$ لما تكون ال Lim بأكثر من متغيرما متقدر نستعمل lopital Rule

7) $\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2}$ $\quad \frac{0}{0}$ بالتعويض المباشر

$\frac{x(x^2-y^2)}{x^2+y^2} \Longrightarrow$ ما نفعتنا بالحل

using polar cooridnates

$$\boxed{x = r\cos\theta, \; y = r\sin\theta \;, \; x^2+y^2 = r^2}$$

$(x,y)\to(0,0)$
$\therefore r \to 0$

$\lim_{r\to 0} \frac{r^3\cos^3\theta - r\cos\theta\, r^2\sin^2\theta)}{r^2}$ $\quad \frac{0}{0}$ بالتعويض المباشر So we can use any Rule

$$\lim_{r\to 0} \frac{\overset{r}{\cancel{r^3}}(\cos^3\theta - \cos\theta \sin^2\theta)}{\cancel{r^2}} \Longrightarrow \lim_{r\to 0} r\,\underbrace{(\cos^3\theta - \cos\theta\, \sin^3\theta)}_{\text{Constant}}$$

$$= 0\,(\text{Constant}) = 0 \Longrightarrow \therefore \lim_{r\to 0} = 0$$ $\therefore$ exists

8) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2y^2}{x^4+y^4}$ بالتعويض المباشر $\frac{0}{0}$ ⟹ We cas use polar Coordinate. , $x,y \to (0,0)$

$\therefore r \to 0$

$$\lim_{r\to 0} \frac{4r^2\cos^2\theta \; r^2\sin^2\theta}{r^4\cos^4\theta + r^4\sin^4\theta} \longrightarrow \lim_{r\to 0} \frac{\cancel{r^4}\,(4\cos^2\theta \, \sin^2\theta)}{\cancel{r^4}\,(\cos^4 + \sin^4\theta)}$$

$$\therefore \lim_{r\to 0} \boxed{\frac{4\,(\cos^2\theta \, \sin^2\theta)}{\cos^4\theta + \sin^4\theta}} \Longrightarrow \text{Constant}$$

θ كل ما تغيرت ⇐ limit تتغير

$\therefore$ DNE ⟹ Since The limit Varies for θ Varies

* Two path test for non Existence of a limit:

⟹ if a function $f(x,y)$ has different limits along two different paths in the Domain of f as $(x,y) \to (x_0, y_0)$

Then;

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ <u>DNE</u>

in most Case: we use the path $y = kx$, $k \in \mathbb{R}$

$y = kx^2$

Example: Show that the function $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ has no limit as $(x,y) \to (0,0)$

Let $y = kx^2$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \frac{2kx^4}{x^4+kx^4} = \lim_{(x,y)\to(0,0)} = \frac{2k\cancel{x^4}}{\cancel{x^4}(1+k)} = \lim_{(0,0)} \frac{2k}{1+k}$$

Let $k = 0$ → $\lim = 0$

$y = 0$

$k = 1$ → $\lim = 1$

$y = x^2$

} Two different limits for diff. Values of k; hence The limits <u>DNE</u>

Remark: having the same limit along all straight lines approaching (x_0, y_0) $\not\Longrightarrow$ Limit exists at (x_0, y_0)

Example: Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{|xy|} = DNE$ The paths: $y = kx$

$\lim\limits_{x\to 0} \frac{x^2k}{|x^2k|} = \lim\limits_{x\to 0} \frac{k\cancel{x^2}}{|k|\cancel{x^2}} = \lim\limits_{x\to 0} \frac{k}{|k|} \longrightarrow |k| = \quad k = 0$

if $k=1, y=x \longrightarrow \lim = 1$
if $k=-1, y=-x \longrightarrow \lim = -1$

$\longrightarrow \lim\limits_{x\to 0} \frac{k}{|k|} = \begin{cases} 1, & k>0 \\ -1, & k<0 \end{cases}$

Hence, two diff. Values with two diff. limits, The limit $\underline{\underline{DNE}}$

Def: A function $f(x,y)$ is Cont. at the point (x_0,y_0) ?

(1) f is defined at (x_0,y_0)

(2) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists (3) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \longrightarrow f(x_0,y_0)$

f is Continuous if it is Cont. at every point of its domain.

example:

① $f(x,y) = \frac{x+y}{x-y}$; is Cont. at all (x,y) Such that $x \neq y$ لم يس

② $f(x,y) = \frac{x+y}{2+\cos x}$; is Cont at all (x,y)

③ $f(x,y,z) = \ln(z - x^2 - y^2 - 1)$; is Cont. at all (x,y,z) Such that $(x^2+y^2+1 < z)$

* Continuity of Composites:

if f is Cont. at (x_0,y_0) and g is a single Variable function Cont. at $f(x_0,y_0)$, Then $h = g\circ f = g(f(x,y))$ is Cont. at (x_0,y_0)

* Functions of more than two Variables:

The definitions of limit and Cont. for function of two Variables and the Conclusions about limits, Continuity for sums, products . . . , all extends to functions of three or more Variables.

* Non - existance (test) :

if different paths ⟹ different limits

Then ⟹ Limit DNE

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ , along } y=x$$

$$y = x^2$$

Monday 11-Nov-2024

14.3:partial derivatives

If $\underset{\downarrow}{z} = \underset{\downarrow}{f(x,y)}$

dep. indep.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y = \lim_{h\to 0}\frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

* Limit exist :

→ derivative = slope of tangent ($z=f(x,y)$)

Slop of tangent at $(x, y_0) = \frac{\partial z}{\partial x}$, slop of tangent at $(x_0, y) = \frac{\partial z}{\partial y}$

example: Find fx by defn. at (4,2) , $f(x) = x^2 + xy$

$$f(x) = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(\overset{x}{4+h},\overset{y}{2}) - f(4,2)}{h} = \lim_{h\to 0}\frac{(4+h)^2 + (4+h)(2) - 24}{h}$$

$$= \lim_{h\to 0}\frac{16+8h+h^2+8+2h-24}{h}$$

$$= \lim_{h\to 0}\frac{h^2+10h+\cancel{24}-\cancel{24}}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(h+10)}{\cancel{h}} \Rightarrow 0+10 = \underline{\underline{10}}$$

If $z = f(x,y)$ then the partial derivative with respect to x is donated by:

$$\frac{\partial f}{\partial x} = f_x = z_x = \frac{\partial z}{\partial x}$$

and the partial derivative with respect to y is donated by:

$$\frac{\partial f}{\partial y} = f_y = z_y = \frac{\partial z}{\partial y}$$

example: find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ at the point $(2,-1)$ for :

1) $f(x,y) = x^2 - xy + y^2$

$f_x = 2x - y \Rightarrow f(2,-1) = 4 + 1 = 5$

$f_y = -x + 2y \Rightarrow f(2,-1) = -2 + (-2) = -4$

2) $f(x,y,z) = xy + yz + xz$, also find $\frac{\partial f}{\partial z}$ at $(2,-1,1)$

$f_x = y + z \Rightarrow f(2,-1,1) = -1 + 1 = 0$

$f_y = x + z \Rightarrow f(2,-1,1) = 2 + 1 = 3$

$f_z = y + x \Rightarrow f(2,-1,1) = -1 + 2 = 1$

Second order partial derivatives:

$Z = f(x,y)$

$f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

example: $f(x,y) = x\cos y + ye^x$, find the 2nd order partial derv.

1st order $f_x = \cos y + ye^x$, $f_y = -x\sin y + e^x$

2nd order $f_{xx} = ye^x$, $f_{xy} = -\sin y + e^x$ $\Rightarrow$ مش دايما بيطلع صح

$f_{yy} = -x\cos y$, $f_{yx} = -\sin y + e^x$

Theorm:

→ if $f(x,y)$ and it's partial der. $(f_x, f_y, f_{xy}, f_{yx}) \Rightarrow$ are all Cont. at point (a,b)

Then $f_{xy} = f_{yx}$ at (a,b)

يعني لما يكونو كلهم متصلين عند (a,b) بكون $f_{xy} = f_{yx}$ بس العكس مش صحيح

example: Find $\underbrace{\frac{\partial^2 w}{\partial x \partial y}}_{w_{yx}}$, $\underbrace{\frac{\partial^2 w}{\partial y \partial x}}_{w_{xy}}$ if $w = xy + \frac{e^y}{y^2+1}$ "The function is Cont."

$w_x = y$, $w_y = x + \frac{(y^2+1)(e^y) - e^y(2y)}{y^2+1}$

Using The Therom:

$w_{xy} = 1 \Rightarrow \therefore w_{yx} = 1$ duo to the function and partial der. are Cont.

example: find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$ $\left(\frac{\partial^2 f}{\partial z \partial y \partial x \partial y} \right)$

اذا غيرنا الترتيب هل بيضل نفس الجواب؟

$f_y = -4yxz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

$f_{xyyz} = ??$

$f_x = -2y^2z + 2xy$

$f_{xy} = -4yz + 2x$

$f_{xyy} = -4z$

$f_{xyyz} = -4$

example: $W = \frac{xy}{y^2 + 2\sin^2 y + 1}$ find W_{yxx}?

by therom $\Rightarrow$ $W_{yxx} = W_{xxy}$

$W_x = \frac{y}{y^2 + 2\sin^2 y + 1}$, $W_{xy} = 0$ $\Rightarrow$ $W_{xyy} = 0$

Theorem:

if a function $f(x,y)$ is differentiable at (x_0,y_0) then f is Cont. at (x_0,y_0)

- Corollary

partial def. f_x & f_y of $f(x,y)$ are Cont. throughout an open region R then f is diff. at every point in R

example: explain why $f(x,y) = 1 + x\ln(xy-5)$ is diff. at $(2,5)$?

$F_x = x\left(\frac{y}{xy-5}\right) + \ln(xy-5) = \frac{xy}{xy-5} + \ln(xy-5)$

$f_y = x\left(\frac{x}{xy-5}\right) = \frac{x^2}{xy-5}$

عشان اثبت انه Cont. لازم الصورة = الـ lim

① f_x is Cont. at $(2,5)$ since : $\lim_{(x,y)\to(2,5)} f_x = f_x(2,5)$

$\lim_{(x,y)\to(2,5)} \frac{xy}{xy-5} + \ln(xy-5) = 2 + \ln 5$

$\therefore f_x$ is Cont. $2+\ln 5 = 2+\ln 5$ ✓

② f_y is Cont. at $(2,5)$ since: $\lim_{(x,y)\to(2,5)} f_y = F_y(2,5)$

$\lim_{(x,y)\to(2,5)} \frac{x^2}{xy-5} = \frac{4}{5}$

$\therefore f_y$ is Cont. $\frac{4}{5} = \frac{4}{5}$ ✓

$\Rightarrow$ from 1 & 2 $f(x,y)$ is diff at $(2,5)$ #

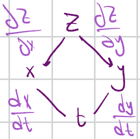
Monday 23-Nov-2024

14.4: chain rule

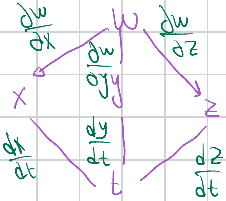
$z = f(x,y)$

$x = g(t),\ y = h(t)$

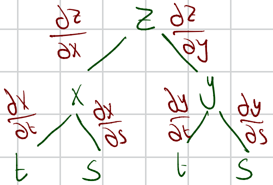
$$\Rightarrow \frac{dz}{dt} = \left(\frac{\partial z}{\partial x} \cdot \frac{dx}{dt}\right) + \left(\frac{\partial z}{\partial y} \cdot \frac{dy}{dt}\right)$$



$w = f(x,y,z)$



$z = f(x,y)\ ,\ x = f(t,s),\ y = f(t,s)$



بعمل partial t
و بتعامل معها
ك Constant

example: $w = xy\ ,\ x = \cos t\ ,\ y = \sin t\ ,$ find $\frac{dw}{dt}$ at $t = \frac{\pi}{2}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= y \cdot -\sin t + x \cdot \cos t$$

$$= -y \sin t + x \cos t$$

$$= -y \sin \frac{\pi}{2} + \cancel{x \cos \frac{\pi}{2}}$$

$$= -y = -\sin \frac{\pi}{2} \Longrightarrow \boxed{\therefore \frac{dw}{dt} = -1}$$

example: $w = xy + z\ ,\ x = \cos t\ ,\ y = \sin t\ ,\ z = t\ ,$ find $\frac{dw}{dt}$ at $t = 0$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= (y \cdot -\sin t) + (x \cdot \cos t) + (1 \cdot 1)$$

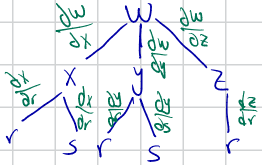
$$= -y \sin t + x \cos t + 1$$

$$= -\underset{0}{\cancel{y \sin 0}} + \underset{1}{\underbrace{x \cos 0}} + 1 \Longrightarrow \boxed{\frac{dw}{dt} = 2}$$

$x = \cos t = \cos 0 = 1$

example: express $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial s}$ in terms of r & s

$w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dr} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dr} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dr}$$

$$= \left(1 \cdot \frac{1}{s}\right) + \left(2 \cdot 2r\right) + \left(2z \cdot 2\right)$$

$$= \frac{1}{s} + 4r + 4z$$

$$= \frac{1}{s} + 4r + \underbrace{4(2r)}_{8r}$$

$$\Longrightarrow \boxed{\frac{\partial w}{\partial r} = \frac{1}{s} + 12r}$$

$$\frac{\partial^2 w}{\partial s\,\partial r} = \frac{-1}{s^2}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{dx}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{dy}{ds}$$

$$= \left(1 \cdot \frac{-r}{s^2}\right) + \left(2 \cdot \frac{1}{s}\right)$$

$$= \frac{-r}{s^2} + \frac{2}{s}$$

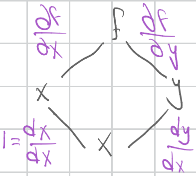
$$\Longrightarrow \boxed{\frac{\partial w}{\partial s} = \frac{-r + 2s}{s^2}}$$

قانون القسمة

$$\frac{\partial^2 w}{\partial s^2} = \frac{2r}{s^3} - \frac{2}{s^2}$$

Implicit Diff.

↳ if $f(x,y) = 0$ find $\frac{dy}{dx}$



$$0 = \frac{\partial f}{\partial x}\cdot\overset{=1}{\boxed{\frac{dx}{dx}}} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\underset{f_x}{\boxed{\frac{\partial f}{\partial x}}}(1) + \underset{f_y}{\boxed{\frac{\partial f}{\partial y}}}\cdot\frac{dy}{dx} = 0$$

$$\Longrightarrow \boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}} \text{ —— } *$$

$f(x,y,z) = 0$, $z = y(x,y)$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}} \quad , \quad \boxed{\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}}$$

example: find $\frac{dy}{dx}$ for $y^2 - x^2 - \sin xy = 0$

Using Calculas I: $2y\left(\frac{dy}{dx}\right) - 2x - \cos(xy)\left(x\frac{dy}{dx} + y(1)\right) = 0$

$$2y\left(\frac{dy}{dx}\right) - 2x - x\cos xy\,\frac{dy}{dx} + y\cos xy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy} \longrightarrow *$$

using therom: $\frac{dy}{dx} = \frac{-f_x}{f_y}$

$$= \frac{-(-2x - y\cos xy)}{2y - x\cos xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy} \longrightarrow *$$

example: $x^3 + z^2 + ye^{xz} = -z\cos y$, Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0,0,0)$

$$x^3 + z^2 + ye^{xz} + z\cos y = 0$$

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} = \left.\frac{-(3x^2 + ye^{xz}(z))}{2z + ye^{x}(x) + \cos y}\right|_{(0,0,0)} \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{0}{1} = 0}$$

$$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z} = \left.\frac{-(e^{xz} - z\sin y)}{2z + ye^{x}(x) + \cos y}\right|_{(0,0,0)} \Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-1}{1} = 1}$$

example: if $z^3 - xz - y = 0$ where z defines as function of x & y find $\frac{\partial^2 z}{\partial x\,\partial y} = z_{yx}$

Using Theorem:

$$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z} = \frac{-(-1)}{3z^2 - x} = \boxed{\frac{1}{3z^2 - x}}$$

$$\frac{\partial z}{\partial x\,\partial y} = \frac{-(-1)}{(3z^2 - x)^2} \Longrightarrow \boxed{\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1}{(3z^2 - x)^2}}$$

Monday 25-Nov-2024

14.5: Directional derivative and gradient vectors

Def: The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit Vector

$\vec{u} = u_1 i + u_2 j$ $\qquad |=|$unit Vector$|$ pix

Directional derivative $$\left(\frac{\partial f}{\partial s}\right)_{u, P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$\left(D_{\vec{u}} f\right)_{P_0}$ → The directional deravative defined by (i) is also denoted by $\underline{\underline{\left(D_{\vec{u}} f\right)_{P_0}}}$

The derivative of f at P_0 in the direction of $\vec{u}$

example: using the def., find the derivative

$f(x,y) = x^2 + xy$ at $P_0(\overset{x_0}{1}, \overset{y_0}{2})$ in the direction of $\vec{u} = \overset{u_1}{\frac{1}{\sqrt{2}}} i + \overset{u_2}{\frac{1}{\sqrt{2}}} j$

$$\left(D_{\vec{u}} f\right)_{P_0} = \lim_{s\to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s}$$

$$= \lim_{s\to 0} \frac{f(\overbrace{1 + \frac{1}{\sqrt{2}}s}^{x}, \overbrace{2 + \frac{1}{\sqrt{2}}s}^{y}) - f(1,2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{1}{\sqrt{2}}s\right)^2 + \left(1 + \frac{1}{\sqrt{2}}s\right)\left(2 + \frac{1}{\sqrt{2}}s\right) - 3}{s}$$

$$\lim_{s\to 0} \frac{\left(1 + \frac{2}{\sqrt{2}}s + \frac{1}{2}s^2\right) + \left(2 + \frac{1}{\sqrt{2}}s + \frac{2}{\sqrt{2}}s + \frac{1}{2}s^2\right) - 3}{s}$$

$$\lim_{s\to 0} \frac{\cancel{3} + \frac{5}{\sqrt{2}}s + s^2 - \cancel{3}}{s} \Rightarrow \lim_{s\to 0} \frac{\cancel{s}\left(\frac{5}{\sqrt{2}} + s\right)}{\cancel{s}}$$

$$\lim_{s\to 0} \frac{5}{\sqrt{2}} + 0 \Longrightarrow \boxed{\frac{5}{\sqrt{2}}}$$

The rate of change at P_0 is Scaller

The rate of change of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction of $\vec{u}$

$$\left(D_{\vec{u}} f\right)_{P_0} = \frac{5}{\sqrt{2}}$$

Def: The gradient Vector of $f(x,y)$ at a point $p_0(x_0,y_0)$ is the Vector

$$\nabla f = \underbrace{\frac{\partial f}{\partial x}}_{f_x} i + \underbrace{\frac{\partial f}{\partial y}}_{f_y} j$$ obtained by evaluating the partial derivatives at $p_0(x_0,y_0)$

Theorem: $$\left(_{\vec{u}}Df\right)_{p_0} = \left(\frac{\partial f}{\partial s}\right)_{u p_0} = (\nabla f)_{p_0} \odot \vec{u}$$ "Scaler Value"

Dot product

Example: $f(x,y) = xe^y + \cos(xy)$, $p_0 = (2,0)$ in the direction of $\vec{u} = 3i - 4j$, Find $\left(_{\vec{u}}Df\right)_{p_0}$

$$\left.\begin{aligned} f_x &= e^y - y\sin(xy) \rightarrow f_x(p_0) = 1 \\ f_y &= xe^y - x\sin(xy) \rightarrow f_y(p_0) = 2 \end{aligned}\right\} \Rightarrow \boxed{\nabla f_{p_0} = i + 2j}$$

$|\vec{u}| = \sqrt{9+16} = 5 \neq 1 \Rightarrow$ not unit Vector

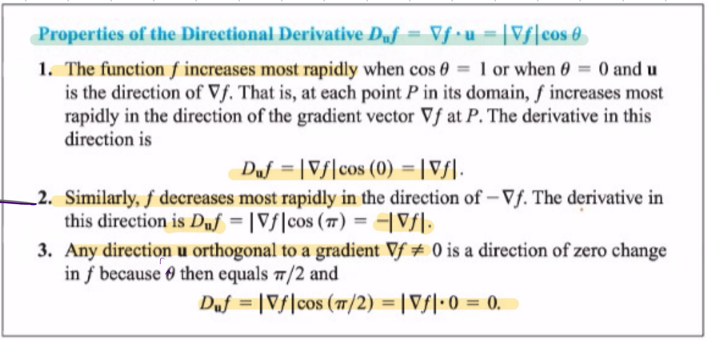
$\vec{v}$ = unit Vector $= \frac{\vec{u}}{|\vec{u}|} = \boxed{\frac{3}{5}i - \frac{4}{5}j}$ → unit Vector

$$\left(_{\vec{u}}Df\right)_{p_0} = \nabla f \cdot \vec{v}$$
$$= (i + 2j) \cdot \left(\frac{3}{5}i - \frac{4}{5}j\right) \Rightarrow \left(_{\vec{u}}Df\right)_{p_0} = \frac{3}{5} - \frac{8}{5} = -1$$

$\left(_{\vec{u}}Df\right)_{p_0} = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$ → θ between $\vec{u}$ & ∇f

$|\vec{u}|$ → unit Vector $|\vec{u}| = 1$

$$\left(_{\vec{u}}Df\right)_{p_0} = \nabla f \cos\theta$$

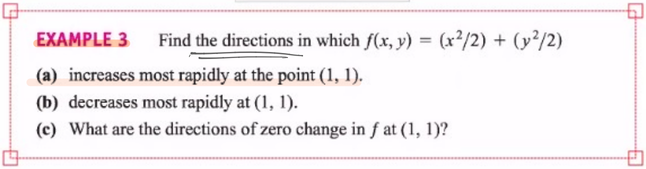


Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

$\cos\theta = -1$

$$\left(_{\vec{u}}Df\right)_{p_0} = |\nabla f|(-1) = -|\nabla f|$$



EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1).

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

a) increase Rapidly in the direction of $\nabla f \Rightarrow$ direction = unit Vector

$\nabla f = xi + yj$ at point $(1,1) \Rightarrow \nabla f = i + j$, $|\nabla f| = \sqrt{2}$

$$\text{direction} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$$

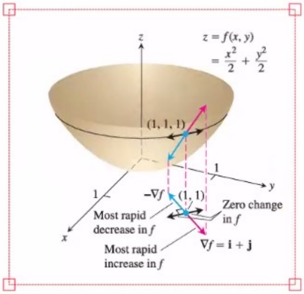
b) the function f decreases most rapidly at $(1,1)$ in the opposite direction of ∇f (or) in the direction of $(-\nabla f)$

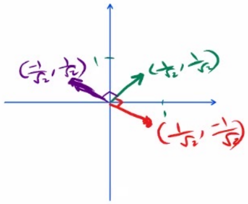
$$\vec{u} = \frac{-\nabla f}{|\nabla f|} = -\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

c) Zero change $\Rightarrow \vec{u} \perp \nabla f$

$$u_1 = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

$$u_2 = \frac{-1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$$





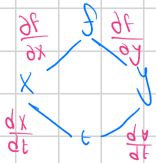
Line : $\vec{u} \parallel$ line , p_0 on Line

$$(Df)_{\vec{u}, p_0} = (\nabla f)_{p_0} \cdot \vec{u}$$

$f(x,y)$

$x = x_0 + t u_1 \Rightarrow \frac{dx}{dt} = u_1$

$y = y_0 + t u_2 \Rightarrow \frac{dx}{dt} = u_2$



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$$\Rightarrow \therefore (Df)_{\vec{u}, p_0} = \frac{df}{dt} = \underbrace{\frac{\partial f}{\partial x} \frac{dx}{dt}} + \underbrace{\frac{\partial f}{\partial y} \frac{dy}{dt}}$$

$$= \left(\frac{\partial f}{\partial x} i + \frac{df}{\partial y} j\right) \cdot \left(\frac{dx}{dt} i + \frac{dy}{dt} j\right)$$

$$\therefore (Df)_{\vec{u}, p_0} = (\nabla f)_{p_0} \cdot \vec{u}$$

Gradients and tangents to level Curve

Level Curve $\Rightarrow f(x,y) = \text{Constant}$

if $f(x,y)$ has a constant C along a smooth Curve

$\vec{r} = g(t)i + h(t)j$, $f(x,y) = C$ level Curve

$f(g(t), h(t)) = c$

$\frac{df}{dt}(g(t), h(t)) = \frac{dc}{dt}$

$X = g(t) \rightarrow \frac{dx}{dt} = g'(t)$

$y = h(t) \rightarrow \frac{dy}{dt} = h'(t)$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t) = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j\right)}_{\nabla f} \cdot \underbrace{\left(g'(t) i + h'(t) j\right)}_{r'(t) = \frac{d\vec{r}}{dt}}$$

∇f is normal to the level Curve

Tangent line equation:

$\Rightarrow$ The line through $p_0(x_0, y_0)$ normal to the Vector

$\vec{N} = Ai + Bj$

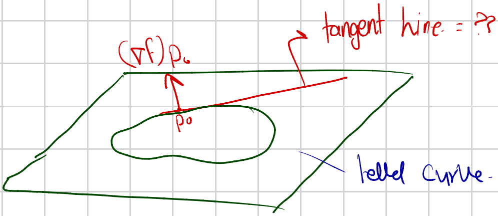
has the equation

$A(x - x_0) + B(y - y_0) = 0$

- if $\vec{N} = (\nabla f)\, p_0$

$Ai + Bj = \underbrace{\frac{\partial f}{\partial x}}_{f_x} i + \underbrace{\frac{\partial f}{\partial y}}_{f_y} j$; $A = f_x$ & $B = f_y$

كيف اوجد معادلة؟



$\therefore$ tangent : $(f_x)_{p_0}(x - x_0) + (f_y)_{p_0}(y - y_0) = 0$ ⟶ *

Example: Find an equation for the tangent to the ellipse:

$\frac{x^2}{4}+y^2=2$ at the point $(\overset{x_0}{-2},\overset{y_0}{1})$

Level curve: $f(x,y)=c$

$\frac{x^2}{4}+y^2=f(x,y)$, $c=2$

$\nabla f = f_x i + f_y j$

$= \frac{x}{2} i + 2y j$, $f_x(-2,1) = \overset{A}{\underline{\underline{-1}}}$, $f_y(-2,1) = \overset{B}{\underline{\underline{2}}}$

⇒ Tangent line equation $A(x-x_0)+B(y-y_0)=0$

$-1(x- -2)+2(y-1)=0$

$-1(x+2)+2(y-1)=0$

$-x-2+2y-2=0$

$2y-x-4=0 \Rightarrow 2y=x+4 \Rightarrow y=\frac{x}{2}+2$

* Function of three Variable

⇒ For a differentiable $f(x,y,z)$ and a unit Vector $u=u_1 i+u_2 j+u_3 k$ in space

We have $\nabla f = f_x i + f_y j + f_z k$

$(_{\vec{u}}Df)_{p_0} = \underline{\underline{\nabla f\cdot\vec{u}}} = f_x u_1 + f_y u_2 + f_z u_3$

↳ $|\nabla f|\underbrace{|\vec{u}|}_{\text{unit Vector}=1}\cos\theta$

then $(_{\vec{u}}Df)_{p_0} = |\nabla f|\cos\theta$

Example: $f(x,y,z)=x^3-xy^2-z$, $p_0(1,1,0)$, find the derivative of f at p_0 in the direction of $\vec{v} = 2i-3j+6k$

a) $|\vec{v}| = \sqrt{2^2+3^2+6^2} = \sqrt{49} = 7 \neq 1$

$\vec{u}$: unit Vector $= \frac{\vec{v}}{|\vec{v}|} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k$

$f_x = 3x^2-y^2 \Rightarrow f_x(p_0)=3-1=2$

$f_y=-2xy \Rightarrow f_y(p_0)=-2$, $f_z=-1$

$(\nabla f)_{p_0}=2i-2j-k$

ii $\left(D_{\vec{u}} f\right)_{p_0} = (\nabla f)_{p_0} \cdot \vec{u}$

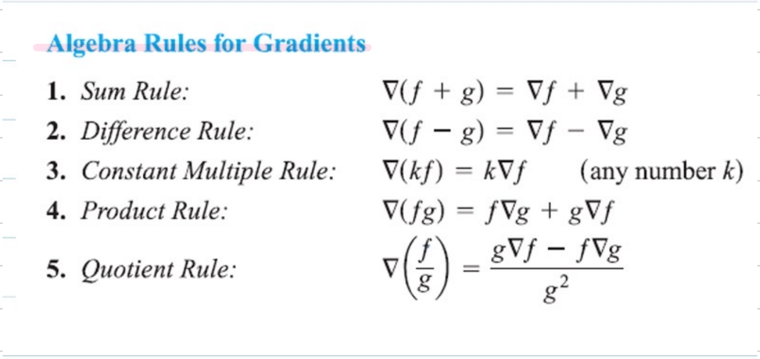
$= 2\left(\frac{2}{7}\right) - 2\left(\frac{-3}{7}\right) - 1\left(\frac{6}{7}\right)$

$\Longrightarrow \left(D_{\vec{u}} f\right)_{p_0} = \frac{4}{7} + \frac{6}{7} - \frac{6}{7}$

$\boxed{\left(D_{\vec{u}} f\right)_{p_0} = \frac{4}{7}}$

b) in what direction does f change most rapidly at p_0 and what are the rates of change in these direction?

- f increases most rapidly in the direction of $\nabla f = 2i - 2j - k$ and the rate of change is $|\nabla f| = \sqrt{9} = 3$
- f decreases most rapidly in the direction of $-\nabla f = -2i + 2j + k$ and the rate of change $-|\nabla f| = -\sqrt{9} = -3$



Algebra Rules for Gradients

1. *Sum Rule:*	$\nabla(f + g) = \nabla f + \nabla g$
2. *Difference Rule:*	$\nabla(f - g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:*	$\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:*	$\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:*	$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Saturday 7-Dec-2024

14.6: Tangent planes and differentials

$\Longrightarrow$ If $\vec{r}(t) = g(t)\,i + h(t)\,j + k(t)\,k$ is a smooth curve on the level surface $f(x,y,z) = C$ of a differentiable function

$$f(g(t),h(t),k(t)) = C$$

$$\frac{d}{dt} f(g(t),h(t),k(t)) = \frac{d}{dt} C$$

$$\frac{\partial f}{\partial g}\frac{dg}{dg} + \frac{\partial f}{\partial h}\frac{dh}{dt} + \frac{\partial f}{\partial k}\frac{dk}{dt} = 0$$

$$\underbrace{(f_x\vec{i} + f_y\vec{j} + f_z\vec{k})}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j} + \frac{dk}{dt}\vec{k}\right)}_{\frac{dr}{dt}} = 0$$

$\Longrightarrow \nabla f \cdot \frac{dr}{dt} = 0$ "∇f is $\perp$ to the curve's velocity vector." $\Rightarrow$ $\overset{\text{normal}}{\nabla f} \perp \overrightarrow{\frac{dr}{dt}}$ tangent vector

* tangent plane at the point $p_0(x_0,y_0,z_0)$ on the level surface $f(x,y,z) = C$ normal to $(\nabla f)_{p_0}$ is

$$f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$$

* Normal line at $p_0(x_0,y_0,z_0)$ parallel to $(\nabla f)_{p_0}$

$x = x_0 + f_x t$, $y = y_0 + f_y t$, $z = z_0 + f_z t$

example: find tangent plane & normal line of the surface

$f(x,y,z) = x^2 + y^2 + z - 9 = 0 \rightarrow z = 9 - x^2 - y^2$ "Paraboloid" at $p_0(1,2,4)$

- $f_x = 2x \Big|_{(1,2,4)} \Rightarrow f_x = 2$
- $f_y = 2y \Big|_{(1,2,4)} \Rightarrow f_y = 4$
- $f_z = 1$

tangent plane using the equation above:

$$2(x-1) + 4(y-2) + 1(z-4) = 0$$

$$2x - 2 + 4y - 8 + z - 4 = 0 \Rightarrow \boxed{2x + 4y + z = 14}$$

normal line using the equations above:

$x = 1 + 2t$, $y = 2 + 4t$, $z = 4 + t$ $\quad t \in \mathbb{R}$

example: find an equation for the plane that is tangent to the surface $z = x\cos y - ye^{x}$ at the point $(0,0,0)$

$f(x,y,z) = x\cos y - ye^{x} - z = 0$

$\nabla f = f_x i + f_y j + f_z k$

$= (\cos y - ye^{x})i + (-x\sin y - e^{x})j - k \Big|_{(0,0,0)} \Longrightarrow \boxed{\nabla f = i - j - k}$

The tangent plane

$\hookrightarrow \quad 1(x-0) - 1(y-0) - 1(z-0) = 0 \Longrightarrow \boxed{x - y - z = 0}$

example: find the parametric equation for the line tangent to the curve of intersection of the surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ at $P_0(1,1,3)$

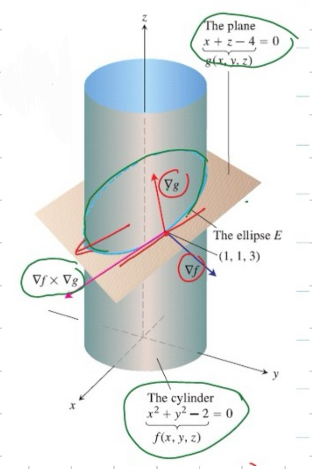
tangent to the ellipse

cylinder

plane

$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$

$= (2x)i + (2y)j \Longrightarrow \boxed{\nabla f(1,1,3) = 2i + 2j}$



$\nabla g = g_x\vec{i} + g_y\vec{j} + g_z\vec{k}$

$= (1)i + 0 + (1)k \Longrightarrow \boxed{\nabla g(1,1,3) = i + k}$

$\vec{v} = \nabla f(1,1,3) \times \nabla g(1,1,3)$ is // to the tangent line.

$$= \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} - 2\vec{k}$$

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$\vec{v}$ // to the tangent line

$\vec{v} = \nabla f \times \nabla g$ at P_0

$\Rightarrow$ The Normal Line:

$x = 1 + 2t \; , \; y = 1 - 2t \; , \; z = 3 - 2t$

* linearization 8

- Two Variables

$\hookrightarrow f(x,y)$ at (x_0, y_0) where f is differentiable, $f(x,y) \approx L(x,y)$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

example: find the linearization of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at point $(3,2)$

$L(x,y) = f(3,2) + f_x(2,3)(x-3) + f_y(2,3)(y-2)$

$= 8 + 4(x-3) - 1(y-2) \Longrightarrow \boxed{L(x,y) = 4x - y - 2}$

$f_x = 2x - y \; , \; f_y = -x + y$

- more than two Variables

Linearization of a function with 3 Variables at a point $P_0(x_0, y_0, z_0)$

$$f(x,y,z) \approx L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$

Example: $f(x,y,z) = x^2 - xy + 3\sin z$ find the linearization of f at $P_0(2,1,0)$

- $f_x = 2x - y \Rightarrow f_x(P_0) = 3$, • $f_y = -x \Rightarrow f_y(P_0) = -2$, • $f_z = 3\cos z \Rightarrow f_z(P_0) = 0$

$f(2,1,0) = 4 - 2 + 3(0) = 2$

$L(x,y,z) = 2 + 3(x-2) - 2(y-1) + 3(z-0)$

$= 2 + 3x - 6 - y + 2 + 3z \Longrightarrow \boxed{L(x,y,z) = 3x - 2y + 3z - 2}$

Find approximate Value of $f(\overset{x}{1.9}, \overset{y}{0.9}, \overset{z}{-0.1})$

⇐ ملاحظ انه النقطة يلي اعطاني اعمل عشانها linearization قريبة من النقطة الاصلية $P_0(2,1,0)$

$f(\underbrace{1.9, 0.9, -0.1}) \approx L(x,y,z)$ at $P_0(2,1,0)$

هدول القيم بعوضهم بدل x, y, z

$= 2(1.9) - 2(0.9) + 3(-0.1) - 2$

$= -0.3$

الايرور مش مطلوب

Recall (Calculus 1) : differentials

$y = f(x)$ function of one Variable

x changes from x_1 to $x_2 \Rightarrow \Delta x = x_2 - x_1 \Rightarrow x_2 = x_1 + \Delta x$

$\Delta f = f(x_2) - f(x_1)$

$\Delta f = f(x_1 + \Delta x) - f(x_1)$ —— (True change) التغير الحقيقي في قيمة f

$\Delta f \approx df = f'(x_1)\Delta x$ $\qquad \overset{= dx}{\Delta x} = x_2 - x_1 =$

$\Delta f \approx df = f'(x_1)dx$ —— in Calculus 1

differentials of a function of two Variables

$f(x,y)$ if we move from (x_0, y_0) to a point $(\overset{x_1}{\underline{x_0 + dx}}, \overset{y_1}{\underline{y_0 + dy}})$

مقدار التغير في $x \Leftarrow dx$

$\Rightarrow df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$ قيمة التغير f

$\searrow \approx \Delta f = f(x_1, y_1) - f(x_2, y_2)$

estimate change in f @ differentials ممكن يطلب بالسؤال

differentials of a function of Three Variables

$(x_0, y_0, z_0) \longrightarrow (x_0 + dx, y_0 + dy, z_0 + dz)$

$$df = f_x(x_0,y_0,z_0)dx + f_y(x_0,y_0,z_0)dy + f_z(x_0,y_0,z_0)dz$$

Total diff = estimate Change

point(1,5)

زاد

example: Cylinder with Radius =1cm and height =5cm, but the Radius Changes by +0.03 and the height Changes by −0.01 نقص, estimate the resulting change in the cylinder Volume.

Volume = $\pi r^2 h$

$f(r,h) = \pi r^2 h$

$f(r) = 2\pi r h$, $f(h) = \pi r^2$

$f(1,5) = 10\pi$, $= \pi$

estimate change = df = $f_r(1,5)\,df + f_h(1,5)\,dh$

$\Rightarrow df = 10\pi(0.03) + \pi(-0.01)$

$= 0.3\pi - 0.1\pi \Rightarrow \boxed{df = 0.2\pi}$

- what is the percentage of estimate chang to original volumee

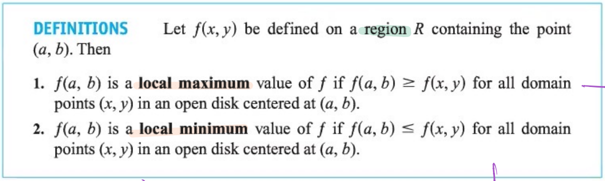
Volume = $\pi(1)(5) \Rightarrow$ Volume = 5π

Percentage = $\frac{0.2\cancel{\pi}}{5\cancel{\pi}}$ ↘ $\frac{\text{estimate}}{\text{الاصلي}}$ ⟹ $\boxed{4\%}$ ⟶ التغير بالنسبة للمساحة الاصلية

Monday 9-Dec-2024

14.7: extreme values and saddle points

⇒ Derivative tests for local extreme Values



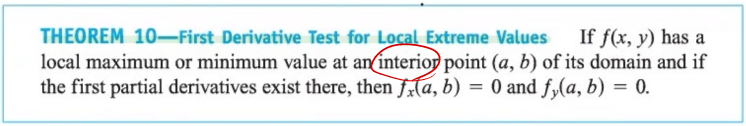
DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

→ صورة أكبر من كل الصور في منطقة حواليها

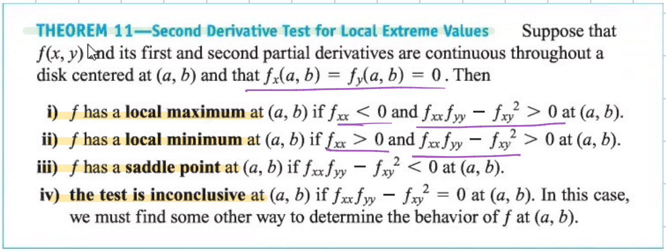
↳ هي اصغر صورة من الصور حواليها

Local = relative



THEOREM 10—First Derivative Test for Local Extreme Values If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

لو انا كان local min or max ⟸ $f_x = 0$ & $f_y = 0$ والعكس غير صحيح



THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

- $f_{xx} f_{yy} - f_{xy}^2 > 0$

$f_{xx} < 0$ → max. $f_{xx} > 0$ → min

- $f_{xx} f_{yy} - f_{xy}^2 < 0$

↳ Saddle Point.

- $f_{xx} f_{yy} - f_{xy}^2 = 0$ → test faild

* Critical point:

↳ an interior point of the domain of a function $f(x,y)$ such that:

f_x and f_y are zero or f_x & f_y DNE

* Saddle point:

↳ A diff. function $f(x,y)$ has a saddle point at (a,b) if in every open disk centered at (a,b) there are points: $f(x,y) > f(a,b)$ and $f(x,y) < f(a,b)$

* Thm: Second derevative test:

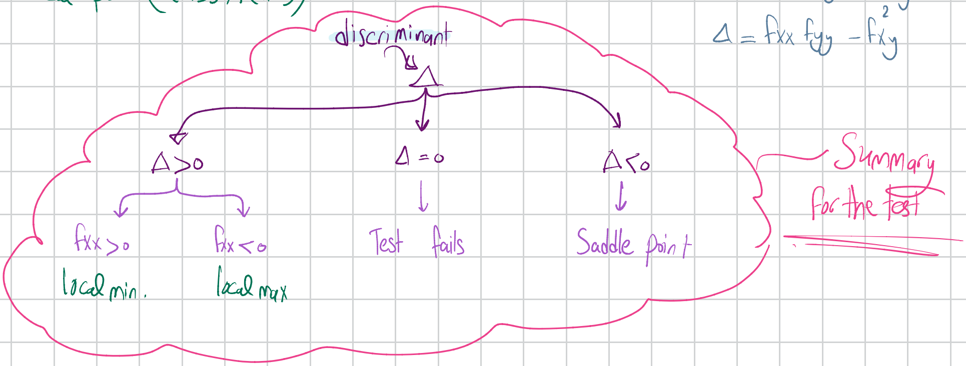
$\boxed{1}$ $f_x = 0$, $f_y = 0$

Critical point $((a,b), f(a,b))$

$\boxed{2}$ f_{xx}, f_{yy}, f_{xy}

$\boxed{3}$ $\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

$\Delta = f_{xx} f_{yy} - f_{xy}^2$



example: find the local extreme values & saddle points of

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

① $f_x = y - 2x - 2 = 0 \Rightarrow y - 2x = 2$ — ① $\Rightarrow \cancel{2y} - 4x = 4$ $\quad -3x = 6$

$f_y = x - 2y - 2 = 0 \Rightarrow x - 2y = 2$ — ② $\quad -\cancel{2y} + x = 2$ $\quad \therefore x = -2$

in equation ①: $y - 2(-2) = 2 \Rightarrow y + 4 = 2 \Rightarrow y = -2$

$\boxed{\text{Critical Value : } (-2, -2)}$

② $f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 1$

$\Delta = f_{xx} f_{yy} - f_{xy}^2$

$= 4 - 1 = \underline{\underline{3 > 0}}$

$((x,y), f(x,y)) = ((-2,2), 8)$

Due to $\Delta > 0$ → $f_{xx} > 0$ / → $f_{xx} < 0$

$f_{xx} = -2 < 0$ So at $(-2, 2)$

There is local max

example: $f(x,y) = e^{2x}\cos y$, find the extreme Values "24"

- $f_x = 2e^{2x}\cos y = 0$, $e^{2x} \neq 0$, $\cos y = 0 \Rightarrow y = \frac{\pi}{2} \mp n\pi$
- $f_y = e^{2x}(-\sin y) = 0$, $e^{2x} = 0$, $\sin y = 0 \Rightarrow y = \mp n\pi$

Not closed & unbounded

$\Rightarrow$ critical points = ?

No critical point : لأنه critical عادةً لازم كل حدود المعادلة تصفر بس هنا e^x مستحيل تصفر

فعشان هيك No critical points

$\Rightarrow$ No critical Values $\rightarrow$ No point where $f_x = f_y = 0$

$\Rightarrow$ No local extreme & No saddle points

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example: find three real numbers whose sum is 9 & whose sum of squers is min

هاي المعلومة يلي بتهمنا

لما يعطي min. or max يكون هو الـ function الأساسي

Let the numbers be $x, y, z \Rightarrow x + y + z = 9 \Rightarrow z = 9 - x - y$

$f(x,y,z) = x^2 + y^2 + z^2$ "Sum of Squers"

$h(x,y) = x^2 + y^2 + (9 - x - y)^2$

$\Rightarrow$ Find the local extreme Values $\Leftarrow$

$h_x = 2x + 2(9 - x - y)(-1)$

$= 2x - 18 + 2x + 2y$

$\Rightarrow h_x = 4x + 2y - 18$ also $h_y = 4y + 2x - 18$

$2x + y - 9$ قسمة 2 $-4y - 2x + 18$ ضرب $(-)$

Then

$$\begin{array}{r} \cancel{2x} + y - 9 \\ + \; \cancel{-2x} - 4y + 18 \\ \hline \end{array} \Rightarrow -3y = 9 \Rightarrow \boxed{y = 3} \; \& \; \boxed{x = 3}$$

Critical point $(3, -3)$

بدي أشوف شو كنها Max or min using Δ

$h_{xx} = 4$

$h_{yy} = 4 \Rightarrow \Delta = 16 - 4 = 12 > 0$

$h_{xy} = 2$ $h_{xx} > 0 \Rightarrow$ w have a local min

There is a local min at (3,3)

Where $x = y = 3 \Rightarrow z = 3$

So The three numbers are $x = 3$, $y = 3$, $z = 3$

* absolute min. & absolute max. on closed bounded region

1) find the critical Values $\Rightarrow f_x = f_y = 0$

↳ interior points

2) find the boundary of the region (الاطراف الفترة المعروف عليها function)

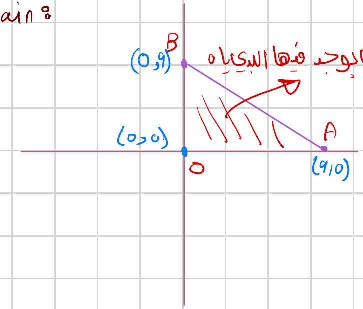
- region → Rectangles, Squares, triangles

example: find absolute max. & min. for

$f(x,y) = 2 + 2x + 2y - x^2 - y^2$, on the triangle region in the first quadrant bounded by

$x=0, y=0$ & $y = 9 - x$

Sketch the domain:



هاي المنطقة يلي يوجد فيها البياناه

① $f_x = 2 - 2x = 0 \Rightarrow x = 1$
$f_y = 2 - 2y = 0 \Rightarrow y = 1$
$\Rightarrow$ Critical point (1,1) ضمن الRegion

② $f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$
$\Delta = 4 > 0 \Rightarrow f_{xx} < 0 \Rightarrow$ There is a local min at (1,1)

هلأ بصير اخد كل قطعة لحال مثل $\bar{AB}$ & $\bar{OA}$ & $\bar{OB}$

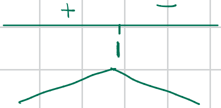
$\bar{OA} \Rightarrow y = 0$, $0 \le x \le 9 \Rightarrow (0,0), (0,9)$ Local min

↳ Two end points

$f(x,y) = f(x,0) = 2 + 2x - x^2$

$f'(x) = 2 - 2x = 0 \Rightarrow x = 1 \Rightarrow (1,0)$

have local max

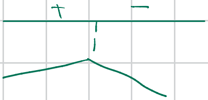


$\bar{OB} \Rightarrow x = 0$, $0 \le y \le 9$

$f(x,y) = f(y) = 2 + 2y - y^2 \Rightarrow (0,0), (9,0) \rightarrow$ two end points

$f'(y) = 2 - 2y = 0 \Rightarrow y = 1 \Rightarrow (0,1)$

have local max



$\overline{AB} \Rightarrow y = 9 - x$

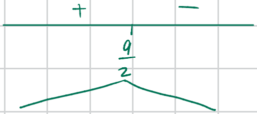
$f(x,y) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$= 2 + 2x + 18 - 2x - x^2 - 81 + 18x - x^2$

$= -2x^2 + 18x - 61 \Rightarrow x=0 \rightarrow (0,9)$ & $y=0 \rightarrow (9,0)$ مكررين

$f'(x) = -4x + 18 = 0 \Rightarrow x = \frac{9}{2}$, $y = 9 - x \Rightarrow y = \frac{9}{2}$

$\Rightarrow \left(\frac{9}{2}, \frac{9}{2}\right)$ have a local max



$f(x,y) = 2 + 2x + 2y - x^2 - y^2$

point (x,y)	$f(x,y)$
$(0,0)$	2
$(1,0)$	3
$(9,0)$	-61
$(0,1)$	3
$(0,9)$	-61
$(1,1)$	4
$\left(\frac{9}{2}, \frac{9}{2}\right)$	$-\frac{41}{2}$

$\Rightarrow$ absolute min $= -61$ at point $(9,0)$ & $(0,9)$

absolute max $= 4$ at point $(1,1)$

Saturday بسم الله الرحمن الرحيم 21-Dec-2024

Chapter 15: Multiple integrals

Ch8 Calculus 1 8 by parts, trigonometric partial fraction

15.1: Double integrals over rectangles

$y = f(x) \Rightarrow \overset{dx\ dy}{f(x,y)}$ defined on a region (R)

↳ rectangle

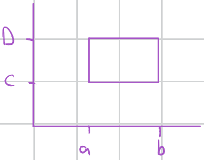
لما اعمل تكامل لاقتران معرف على فترة ⇐ area و لما اعمل تكامل لـ region بعطيني Volume

The Double integral of f over R, is:

$$\int_c^D \int_a^b f(x,y)\,dx\,dy \quad \text{or} \quad \int_a^b \int_c^D f(x,y)\,dy\,dx$$

Thm: if $f(x,y)$ is Cont. throughout a rectangular Region (R)

$a \leq x \leq b$, $c \leq y \leq D$



Then:
$$\iint_R f(x,y)\,dA = \int_c^D \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^D f(x,y)\,dy\,dx$$

example: $f(x,y) = 100 - 6x^2y$ Calculate $\iint_R f(x,y)\,dA$

$R: \; 0 \leq x \leq 2$, $-1 \leq y \leq 1$

$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_0^2 (100 - 6x^2y)\,dx\,dy$$

$$= \int_{-1}^{1} \left(100x - \frac{6x^3}{3}y\right)\Big|_0^2 dy \Rightarrow \int_{-1}^{1} 200 - 16y\ dy$$

كاملنا بالنسبة لـ x وعوضنا الارقام وهذا متكامل بالنسبة لـ y و بعوض

$$\int_{-1}^{1} 200 - 16y\,dy = 200y - \frac{16y^2}{2}\Big|_{-1}^{1} = (200 - \cancel{8}) - (-200 - \cancel{8})$$

$$= \underline{\underline{400}}$$

حجم

$$V = \iint f(x,y)\,dA$$

↘ dxdy or dydx مساحة

If $f(x,y) = 1 \Rightarrow \iint dA = \text{Area}$

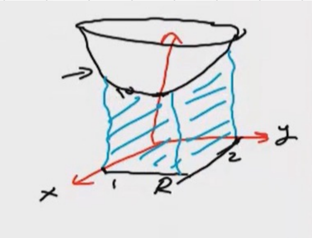
example: find the Volume of the region bounded above by $z = 10 + x^2 + 3y^2$ and below the Rectangle R

Paraboloid

$0 \leq x \leq 1 \quad , \quad 0 \leq y \leq 2$

بدون ما يحكي انها Rectangle

من x & y "حدودهن ثابتة" كلهن يعني Rectangle



$$V = \int_0^2 \int_0^1 (10 + x^2 + 3y^2) \, dx \, dy$$

$$= \int_0^2 \left[10x + \frac{x^3}{3} + 3y^2 x \right]_0^1 dy$$

$$= \int_0^2 10 + \frac{1}{3} + 3y^2 \; dy$$

$$= \left[10y + \frac{1}{3}y + y^3 \right]_0^2 = 20 + \frac{2}{3} + 8$$

$$V = \frac{86}{3}$$

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Example: $I = \iint_R xy e^{xy^2} \, dA \qquad 0 \leq x \leq 2 \; , \; 0 \leq y \leq 1$ rectangle

← دايماً بختار التكامل الاسهل عشان احله →

هون اذا بدي اكامل بالنسبة لـ x بكون
صعب ولازم اكامل by parts اما اذا بلشت
بـ y بكون التكامل اسهل

$$= \int_0^2 \left(\frac{1}{2} \int_0^1 (2xy e^{xy^2} dy) \right) dx$$

$$= \int_0^2 \left[\cancel{2xy} \frac{e^{xy^2}}{\cancel{2xy}} \; dx \right]_0^1$$

$$= \int_0^2 \frac{1}{2} e^{x^2} - \frac{1}{2} e^0 \; dx$$

$$= \int_0^2 \frac{e^{x^2}}{2} - \frac{1}{2} \, dx \Rightarrow \left[\frac{e^x}{2} - \frac{1}{2}x \right]_0^2$$

$$= \left(\frac{e^2}{2} - 1 \right) - \left(\frac{1}{2} - 0 \right)$$

$$= \frac{e^2}{2} - 1 - \frac{1}{2} \Rightarrow \frac{e^2}{2} - \frac{3}{2}$$

Monday 6-Jan-2025

15.2:Double integrals over general region

Thm: let $f(x,y)$ be a Cont. on a region R

1) if $R: a \leq x \leq b , g_1(x) \leq y \leq g_2(x)$ عبارة y بدلالة x $g_1 \& g_2$ Cont. on $[a,b]$

Then: $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

المعرفة بدلالة متغيرات بحدودها جوا و المعرفة بدلالة
ثوابت بحدودها برا

2) if $R: c \leq y \leq d , h_1(y) \leq x \leq h_2(y)$

Then: $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Example Find the Volume of the prism (منشور) whose base is the triangle in xy-plane bounded by x-axis & $y=x$, $x=1$ & whose top lies on the plane $z=f(x,y)=3-x-y$

in plane z قمة المنشور موجود

$$v = \iint_R f(x,y)\,dA$$

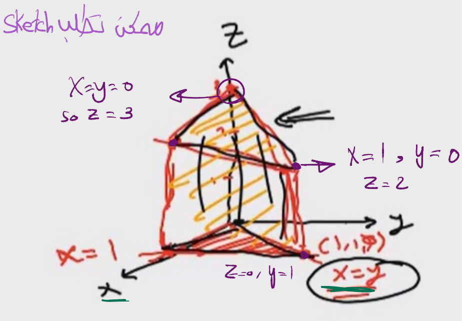
$$= \int_0^1 \int_0^x (3-x-y)\,dy\,dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 3x - x^2 - \frac{x^2}{2}\,dx$$

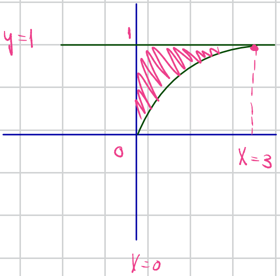
$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} - \frac{x^3}{6}\right]_0^1$$

$$= \left[\frac{3(3)}{2(3)} - \frac{1(2)}{3(2)} - \frac{1}{6}\right. \rightarrow \frac{9}{6} - \frac{2}{6} - \frac{1}{6} = \frac{6}{6} = \underline{\underline{1}}$$



X axis & $y=x$ القاعدة محصورة بين

example: $I=\int_0^3\int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\,dy\,dx$



$y=\sqrt{\frac{x}{3}}$

$x=3y^2$

$$=\int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy$$

$$=\int_0^1 x e^{y^3}\Big]_0^{3y^2} dy$$

عشان التكامل يصير اسهل منبلش نكامل ب x
ثبتنا y من (٠ ← ١) و X هي اللي بتتغير

$$=\int_0^1 3y^2 e^{y^3}\,dy$$

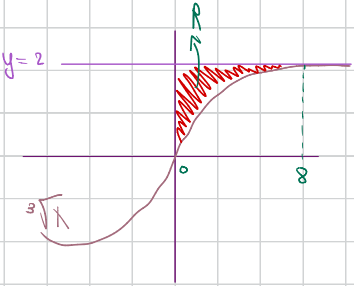
$$=\Big[\frac{\cancel{3}y^2 e^{y^3}}{\cancel{3y^2}} = e^{y^3}\Big]_0^1 = e^1-e^0 = e-1$$

H.W: $\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$, Sketch the region of integration then find the double integral.

$y=\sqrt[3]{x} \Rightarrow x=y^3$

$$\int_0^2\int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$$

$$\int_0^2 \frac{x}{y^4+1}\Big]_0^{y^3} dy$$



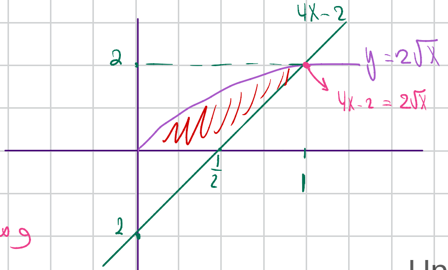
$$\frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\,dy \rightarrow$$

بضرب ب $\frac{4}{4}$ عشان احكي البسط
مشتقة المقام

$$\frac{1}{4}\Big[\ln(y^4+1)\Big]_0^2 = \frac{1}{4}\ln 16+1 - \cancel{\frac{1}{4}\ln 1} = \frac{\ln 17}{4}$$

example: find the volume of the solid that lies beneath the surface $z=16-x^2-y^2$ and above the Region R bounded by $y=2\sqrt{x}$, $y=4x-2$ & x axis.

$x=\frac{y^2}{4}$, $x=\frac{y+2}{4}$



$4x-2=2\sqrt{x} \rightarrow$ وبعد ما نحل المعادلة
بطلع $X=1$

$$V=\int_0^2\int_{\frac{y^2}{4}}^{\frac{y+2}{4}} (16-x^2-y^2)\,dx\,dy$$

ومنحل التكاملات عادي زي قبل

Wednesday 8-Jan-2025

15.3: area by double integrals

Def.: The area of the closed & bounded region is

$$\text{Area of } R = \iint_R dA \rightarrow f(x,y) = 1$$

example: find the area of R enclosed by the parabola $y = x^2$ & the line $y = x + 2$

$$\text{area} = \iint_R dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx$$

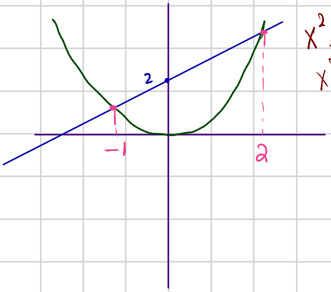
$$= \int_{-1}^{2} y \Big]_{x^2}^{x+2} dx = \int_{0}^{2} x + 2 - x^2\,dx$$

$$= \Big[\frac{x^2}{2} + 2x - \frac{x^3}{3} \Big]_0^2$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \Big(\frac{1}{2} - \frac{2(2)}{1(2)} + \frac{1}{3}$$

$$= 6 - \frac{8}{3} + \frac{3}{4} - \frac{1}{3} = 6 + \frac{3}{4} - \frac{9}{3}$$

$$\Rightarrow \frac{(4)6}{(4)1} + \frac{3}{4} - \frac{3(4)}{1(4)} = \frac{24 + 3 - 12}{4} = \underline{\underline{\frac{15}{4}}}$$



$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ & $x = -1$

* average Value of (f) over R

$$= \frac{1}{\text{area}} \iint_R f\,dA \quad \Rightarrow \quad \frac{\text{Volume}}{\text{area}} = \text{The average Value.}$$

Volume

$\iint dA$

Example: find the average Value for the function $f(x,y) = x\cos xy$ over the region R

$0 \leq x \leq \pi$

$0 \leq y \leq 1$

$$1)\ V = \int_0^{\pi}\int_0^{1} x\cos xy\,dy\,dx$$

$$= \int_0^{\pi} \sin xy \Big|_0^1 dx \Rightarrow \int_0^{\pi} \sin x\,dx \Rightarrow \Big[-\cos x \Big]_0^{\pi} = -\cos\pi - -\cos 0$$

$$= - -1 + 1 = \underline{\underline{2}}$$

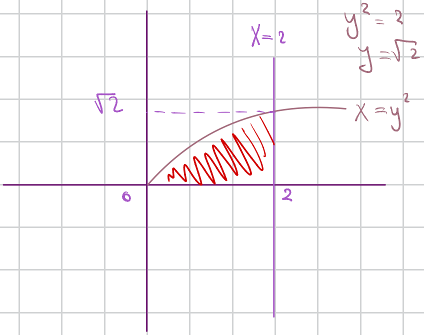
2) area $= \int_0^{\pi}\int_0^{1} dy\,dx$

$= \int_0^{\pi} y\Big]_0^1 dx \quad = \int_0^{\pi} 1\,dx \quad = \Big[x\Big]_0^{\pi} \quad = \underline{\underline{\pi}}$

3) average Value $= \dfrac{\text{Volume}}{\text{area}} = \dfrac{2}{\pi}$ #

example: find the average Value of $f(x,y) = 12y^2e^{x^3}$ over the region R $y^2 \leq x \leq 2$, $0 \leq y \leq \sqrt{2}$

$x = y^2$, $x = 2$



1) $V = \int_0^2\int_0^{\sqrt{x}} 12y^3 e^{x^3}\,dy\,dx$

$= \int_0^2 3y^4 e^{x^3}\,dx\Big]_0^{\sqrt{x}}$

$= \int_0^2 3x^2 e^{x^3}$

let $u = x^3$

$du = 3x^2\,dx$

$x = 0 \rightarrow u = 0$

$x = 2 \rightarrow u = 8$

$= \int_0^8 e^u\,du$

$= \Big[e^u\Big]_0^8 \Rightarrow e^8 - 1 = \text{Volume}$

2) area $= \int_0^2\int_0^{\sqrt{x}} dy\,dx$

$= \int_0^2 y\,dx\Big]_0^{\sqrt{x}}$

$= \int_0^2 \sqrt{x}\,dx \quad = \int_0^2 (x)^{\frac{1}{2}}$

$= \Big[\frac{2}{3}x^{3/2}\Big]_0^2$

area $= \dfrac{4\sqrt{2}}{3}$

3) average Value

$= \dfrac{\text{Volume}}{\text{area}}$

$= \dfrac{e^8 - 1}{\frac{4\sqrt{3}}{3}}$

average Value $= \dfrac{3}{4\sqrt{2}}\left(e^8 - 1\right)$

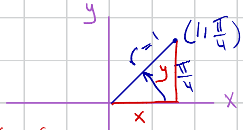
Saturday

11-Jan-2025

15.4: double integral in polar form

Recall from calculus 2:

$(r,\theta) = (1, \frac{\pi}{4})$

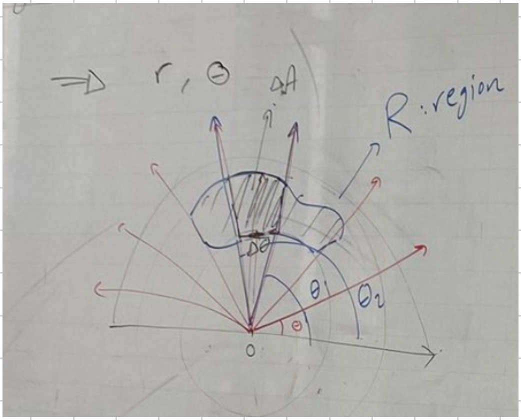


هاي النقطة عبارة
Polar coordinate is

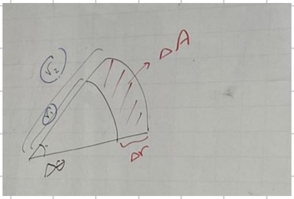
$x = r\cos\theta$

$y = r\sin\theta$ | $\frac{y}{x} = \tan\theta$

$x^2 + y^2 = r^2$



$\overset{?}{\iint f(x,y)\,\boxed{dA}}$ $x,y \rightarrow r,\theta$

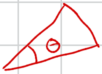


$\Delta A = A_2 - A_1$

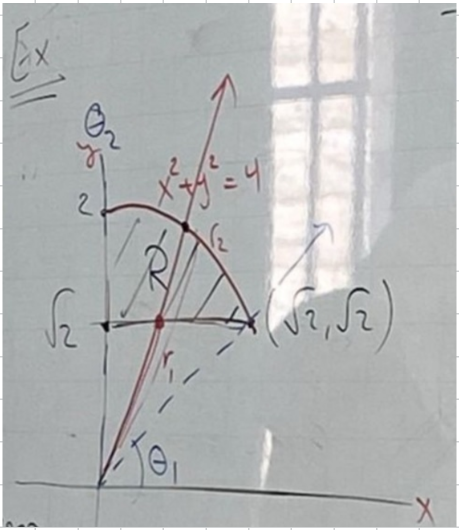
↳ $dA = dx\,dy = dy\,dx$

$dA = r\,dr\,d\theta$

$A = \frac{\cancel{\pi} r^2 \theta}{2\cancel{\pi}} = \frac{1}{2}\theta r^2$



example:



- $y = \sqrt{2} \Longrightarrow r\sin\theta = \sqrt{2} \Longrightarrow$ ن $r = \frac{\sqrt{2}}{\sin\theta} = \sqrt{2}\csc\theta$
- $x^2 + y^2 = r^2 = 4$; $r = 2$

$$v = \iint_R f(x,y)\,dA$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2}\csc\theta}^{2} f(r\cos\theta, r\sin\theta)$$

$V = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$ —— Volume

$A = \iint_R dA = \iint_R r\,dr\,d\theta$ —— area

average Value of f over R $= \frac{V}{A}$

example: find the limits of integration for $f(r,\theta)$ over R that lies inside the Cardioid $r = 1+\cos\theta$ & outside the circle $r=1$

$1 \leq r \leq 1+\cos\theta$

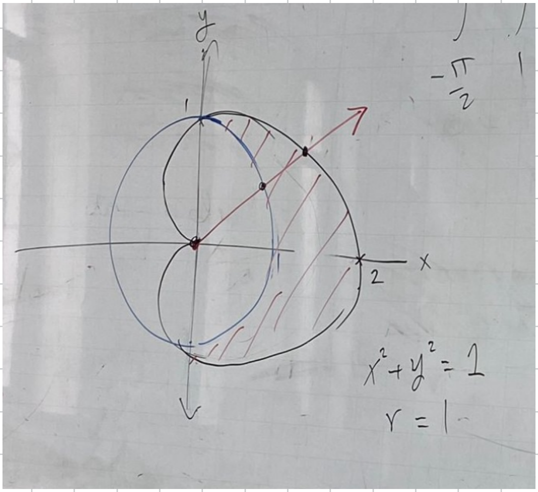
$\rightarrow \theta = 1+\cos\theta = 1$

$\cos\theta = 0$

$\theta = \pm\frac{\pi}{2}$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$



example: $I = \iint_R e^{x^2+y^2}\,dy\,dx$; where R is semicircle region bounded by x-axis & $y = \sqrt{1-x^2}$ → $x^2+y^2=1$

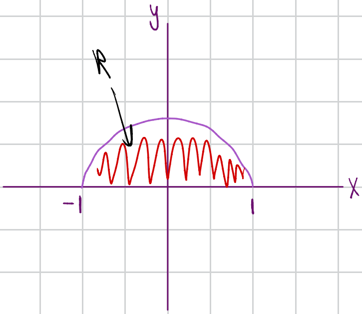
$r^2 = 1 \Rightarrow r = 1$

$$I = \int_0^{\pi} \frac{1}{2} \int_0^1 2 e^{r^2}\, r\,dr\,d\theta$$

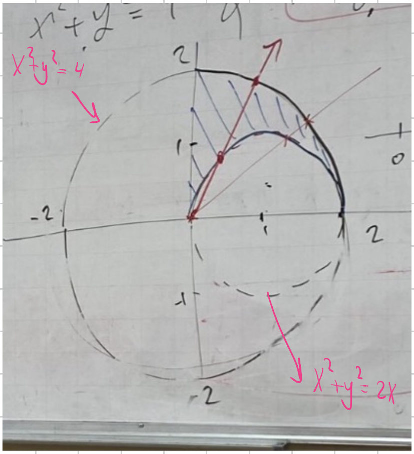
$$= \int_0^{\pi} \frac{1}{2} \left[\frac{\cancel{2}r e^{r^2}}{\cancel{2r}} \right]_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{2}e^1 - \frac{1}{2}e^0\,d\theta = \int_0^{\pi} \frac{1}{2}(e-1)\,d\theta$$

$$= \Big[\theta\,\frac{1}{2}(e-1)\Big]_0^{\pi} \Rightarrow \boxed{I = \frac{\pi}{2}(e-1)}$$



example: $\iint_R y\,dA$, where R lies in the first quadrant lies between $x^2+y^2=4$ & $x^2+y^2=2x$



$x^2-2x+1+y^2=0+1$ بكمل المربع

$(x-1)^2+y^2=1$

↳ circle with center $(1,0)$

radius $=1$

$(x-1)^2+y^2=1$

$x^2+y^2=2x$

$\cancel{r^2}=2\cancel{r}\cos\theta \Rightarrow \boxed{r=2\cos\theta}$

$x^2+y^2=4 \Rightarrow r^2=4 \Rightarrow \boxed{r=2}$

$$V=\int_0^{\pi/2}\int_{2\cos\theta}^{2} r\sin\theta\, r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_{2\cos\theta}^{2} r^2\sin\theta\,dr\,d\theta \Longrightarrow \text{answer}=2$$

وبعد حل التكامل

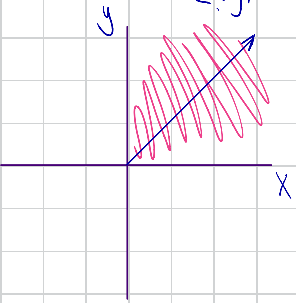
example: use your knowlage in 15.4 to prove $I=\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

$$I^2=\left(\int_0^\infty e^{-x^2}dx\right)\cdot\left(\int_0^\infty e^{-y^2}dy\right)$$

$$=\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}dx\,dy$$

$$=\int_0^\infty\int_0^\infty e^{-x^2-y^2}dx\,dy \qquad r^2=x^2+y^2$$

الربع الأول



$0\le y\le\infty$

$0\le x\le\infty$

$$=\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left(\lim_{A\to\infty}\int_0^A -\frac{1}{2}2e^{-r^2}r\,dr\right)d\theta=\int_0^{\pi/2}\left(\lim_{A\to\infty}-\frac{1}{2}e^{-r^2}\right)_0^A d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\lim_{A\to\infty}-\frac{1}{2}\cancel{e^{-A^2}}+\frac{1}{2}\right.\Longrightarrow\int_0^{\pi/2}\frac{1}{2}d\theta=\left[\frac{\theta}{2}\right]_0^{\pi/2}\Longrightarrow\frac{\pi}{4}-0=\frac{\pi}{4}=I^2$$

0

$A\to\infty$

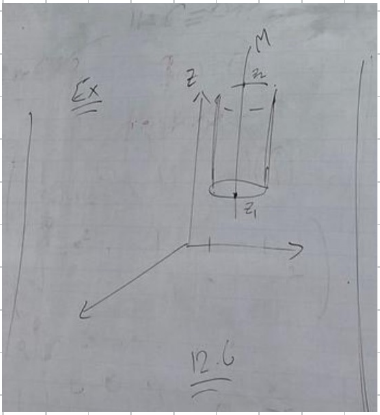
So $I=\frac{\sqrt{\pi}}{2}$ #

Monday 13-Jan-2025

15.5: triple integral in rectangular coordinates

$V = \iiint_D dV \leftarrow dz\,dy\,dx \Longrightarrow V = \iiint_D f(x,y,z)\,dv$

↳ The Volume of a closed bounded region



example: evaluate $I = \int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\,dz\,dy\,dx$

$$I = \int_0^1 \int_0^{1-x^2} xz \Big|_3^{4-x^2-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x^2} x(4-x^2-y) - x(3)\;dy\,dx$$

$$= \int_0^1 \int_0^{1-x^2} x - x^3 - xy\;dy\,dx$$ Double integral

$$= \int_0^1 xy - yx^3 - \frac{xy^2}{2} \Big]_0^{1-x^2} dx \Longrightarrow \int_0^1 x(1-x^2) - x^3(1-x^2) - \frac{x}{2}(1-x^2)\;dx$$

→ ونحل عادي زي قبل ←