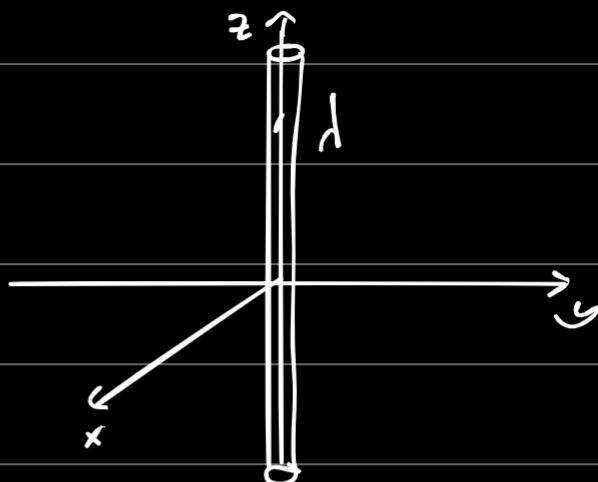


Example: write the charge density of a uniformly charge rod that lies along the z-axis

$$\rho(\vec{r}) = \lambda \delta(x) \delta(y) \leftarrow$$

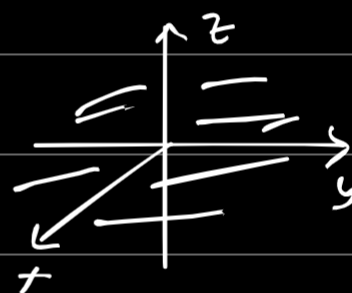
$$\rho(\vec{r}) = \underset{\uparrow}{C} \delta(s) \leftarrow$$

$$= \frac{\lambda}{s} \delta(s)$$



Example: Uniform ^{Surface} charge density on the xy-plane (σ)

$$\boxed{\rho(\vec{r})} = \sigma \delta(z) \quad \text{Cartesian}$$



$$\rho(\vec{r}) = \frac{C}{r^2 \sin \theta} \delta(\theta - \frac{\pi}{2})$$

spherical coordinates

Example. Find $I = \int (r^2 + 1) \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) d\tau$ over a sphere of radius R

Method 1:

$$\vec{\nabla} \cdot \left((r^2 + 1) \frac{\hat{r}}{r^2} \right) = \vec{\nabla} (r^2 + 1) \cdot \frac{\hat{r}}{r^2} + (r^2 + 1) \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right)$$

$$\Rightarrow I = \int \vec{\nabla} \cdot \left((r^2 + 1) \frac{\hat{r}}{r^2} \right) d\tau - \int \vec{\nabla} (r^2 + 1) \cdot \frac{\hat{r}}{r^2} d\tau$$

$$I = \oint \frac{(r^2 + 1)}{r^2} \hat{r} \cdot d\vec{a} - \int 2r \hat{r} \cdot \frac{\hat{r}}{r^2} d\tau$$

$$I = \int_0^{\pi} \int_0^{2\pi} \frac{R^2 + 1}{R^2} \cdot \cancel{R^2} \sin\theta d\theta d\phi - \int_0^{\pi} \int_0^{2\pi} \int_0^R \frac{2r}{r^2} \cancel{r^2} \sin\theta d\phi d\theta dr$$

$$= 4\pi \cdot (R^2 + 1) - R^2 4\pi$$

$$= 4\pi$$

Method 2: $I = \int (r^2 + 1) \underbrace{\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right)}_{4\pi \delta(\vec{r})} d\tau$

$$I = \int \underbrace{(r^2 + 1)}_{= 4\pi (r^2 + 1)} 4\pi \delta(\vec{r}) d\tau \quad \checkmark$$

$$= 4\pi (r^2 + 1) \Big|_{r=0} = 4\pi$$

Helmholtz Theorem:

$$\vec{\nabla} \cdot \vec{F} = \vec{C}$$

$$\vec{\nabla} \times \vec{F} = \vec{D}, \quad \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

and $\vec{F}$ is specified on the boundaries,

then $\vec{F}$ is uniquely determined everywhere

In particular, if $\vec{D}$ and $\vec{C}$ vanishes as $r \rightarrow \infty$ faster than $\frac{1}{r^2}$, then $\vec{F}$ is given by

$$\vec{F} = -\frac{1}{4\pi} \vec{\nabla} \left[\int \frac{\vec{\nabla}' \cdot \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \right] + \frac{1}{4\pi} \vec{\nabla} \times \left[\int \frac{\vec{\nabla}' \times \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \right]$$

$$\vec{F} = -\vec{\nabla} V(\vec{r}) + \vec{\nabla} \times \vec{A}(\vec{r})$$

Recall, $\vec{E} = -\vec{\nabla} V$ ← scalar potential
 $\vec{B} = \vec{\nabla} \times \vec{A}$ ← vector potential

Theorem 1: if $\vec{F}$ is irrotational
(i.e. $\vec{\nabla} \times \vec{F} = 0$), then

① $\vec{F}$ can be written as a gradient
of a scalar function

$$\vec{F} = -\vec{\nabla} V$$

② $\int \vec{F} \cdot d\vec{l}$ is independent of path

$$\oint \vec{F} \cdot d\vec{l} = 0$$

$$\vec{\nabla} \times \vec{F} = 0$$

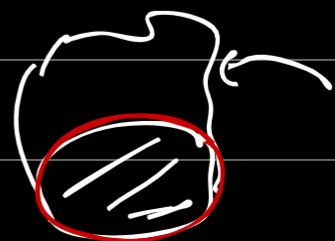
e.g. $\oint \vec{F} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{a} = \int \vec{0} \cdot d\vec{a} = 0$

↑
Stokes
Theorem

Theorem 2: if $\vec{F}$ is divergenceless (solenoidal)
then

$$\vec{\nabla} \cdot \vec{F} = 0$$

② $\int \vec{F} \cdot d\vec{a}$ is independent of the surface
for given boundary line



$$\textcircled{3} \oint \vec{F} \cdot d\vec{a} = 0$$



$\textcircled{4}$ $\vec{F}$ can be written as a curl of a vector field i.e. $\vec{F} = \vec{\nabla} \times \vec{A}$

$$\text{e.g. } \oint \vec{F} \cdot d\vec{a} = \int \underbrace{\vec{\nabla} \cdot \vec{F}}_{\substack{\text{Divergence} \\ \text{Theorem}}} d\tau = \int \underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})}_{0} d\tau = 0$$

Example: uniform charge density on
a sphere of radius R

$$\sigma = \frac{Q}{4\pi R^2}$$

$$\rho(\vec{r}) = k_1 \delta(x^2 + y^2 + z^2 - R^2)$$

$$\rho(\vec{r}) = k_2 \delta(\sqrt{x^2 + y^2 + z^2} - R)$$

$\nearrow \sigma$

$$\int_{\text{all space}} \rho(\vec{r}) d\tau = Q$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k_2 \delta(\sqrt{x^2 + y^2 + z^2} - R) dx dy dz = Q$$

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \end{aligned} \rightarrow \int_0^\infty \int_0^\pi \int_0^{2\pi} k_2 \delta(r - R) r^2 \sin\theta d\phi d\theta dr$$
$$= 4\pi k_2 \int_0^\infty r^2 \delta(r - R) dr$$

$$= 4\pi k_2 R^2 = Q \Rightarrow k_2 = \sigma$$

$$\int_0^{2\pi} \int_0^\pi \sin\theta d\theta d\phi = 4\pi$$

Example: Show that

$$\int_V \vec{\nabla} \psi d\tau = \oint_S \psi d\vec{a}$$

constant
but arbitrary
vector

$$\vec{c} \cdot \oint_S \psi d\vec{a} = \oint_S \vec{c} \cdot (\psi d\vec{a}) = \oint_S \psi \vec{c} d\vec{a}$$

$$= \int_V \vec{\nabla} \cdot (\psi \vec{c}) d\tau$$

$$= \int_V [\vec{\nabla} \psi \cdot \vec{c} + \psi \underbrace{\vec{\nabla} \cdot \vec{c}}_0] d\tau$$

$$= \int_V \vec{c} \cdot \vec{\nabla} \psi d\tau = \underline{\vec{c} \cdot \int_V \vec{\nabla} \psi d\tau}$$

$\therefore$ since $\vec{c}$ is an arbitrary vector
we conclude $\oint_S \psi d\vec{a} = \int_V \vec{\nabla} \psi d\tau$

Area of a surface $A = \int_S da$

Also, we can define "vector Area"

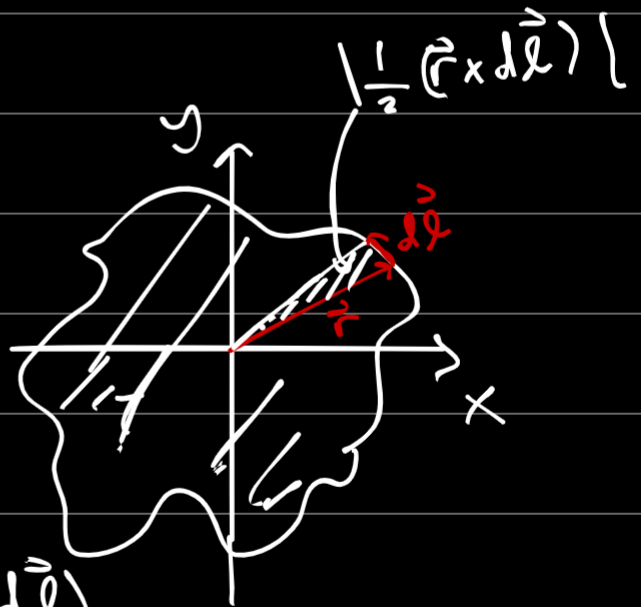
$$\vec{A} = \int_S d\vec{a} = \int_S \hat{n} da$$

Note: ① $\oint_S d\vec{a} = 0$ $\vec{c}$ is a constant vector

$$\vec{c} \cdot \oint_S d\vec{a} = \oint_S \vec{c} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{c} dV = 0$$

Divergence theorem

$$\textcircled{2} \underline{\vec{A} = \frac{1}{2} \oint \vec{r} \times d\vec{l}}$$



$$\vec{c} \cdot \oint \vec{r} \times d\vec{l} = \oint \vec{c} \cdot (\vec{r} \times d\vec{l})$$

$$= \oint (\vec{c} \times \vec{r}) \cdot d\vec{l}$$

$$= \int_S \vec{\nabla} \times (\vec{c} \times \vec{r}) \cdot d\vec{a} = \int_S 2\vec{c} \cdot d\vec{a} = 2\vec{c} \cdot \int_S d\vec{a}$$

$$\vec{\nabla} \times (\vec{c} \times \vec{r}) = (\vec{r} \cdot \vec{\nabla}) \vec{c} - (\vec{c} \cdot \vec{\nabla}) \vec{r} + \vec{c} \underbrace{\vec{\nabla} \cdot \vec{r}}_3 - \vec{r} \underbrace{\vec{\nabla} \cdot \vec{c}}_0$$

$$= 0 - \vec{c} + 3\vec{c} - 0 = 2\vec{c}$$

$$(\vec{c} \cdot \vec{\nabla}) \vec{r} = \left[c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z} \right] [x\hat{x} + y\hat{y} + z\hat{z}]$$

$$= c_x \hat{x} + c_y \hat{y} + c_z \hat{z} = \vec{c}$$

$$\therefore 2\vec{A} = \oint \vec{r} \times d\vec{r} \Rightarrow \vec{A} = \frac{1}{2} \oint \vec{r} \times d\vec{r}$$

$$* (\vec{r} \cdot \vec{\nabla}) \vec{c} = [(x\hat{x} + y\hat{y} + z\hat{z}) \cdot (\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z})]$$

$$[c_x \hat{x} + c_y \hat{y} + c_z \hat{z}]$$

$$= \left[x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right] [c_x \hat{x} + c_y \hat{y} + c_z \hat{z}]$$

$$= 0$$