Chapter 3: Differentiation

1. Find the derivatives of the following functions

a) $f(x) = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

Solution:

$$f'(x) = \frac{[\sqrt{x}+1]\left[\frac{1}{2\sqrt{x}}\right] - [\sqrt{x}-1] \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

$$= \frac{\frac{1}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{2} + \frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

$$= \frac{\frac{1}{\sqrt{x}}}{(\sqrt{x}+1)^2} = \boxed{\frac{1}{\sqrt{x}\,[\sqrt{x}+1]^2}}$$

b) $f(x) = \left(\frac{1}{x} - x\right)(x^2+1)$

Solution:

$$f(x) = x + \frac{1}{x} - x^3 - x$$

$$f(x) = \frac{1}{x} - x^3$$

$$\boxed{f'(x) = \frac{-1}{x^2} - 3x^2}$$

(c) $g(x) = \sec(2x+1)\cot x^2$

Solution:

$$g'(x) = \sec(2x+1) \cdot -(\csc^2 x^2) \cdot 2x + \cot x^2 \left(\sec(2x+1)\cancel{\tan 2}\right)\left(\tan 2x+1\right) \cdot 2$$

$$= \left[\sec(2x+1)\right]\left[-2x \csc^2 x^2 + 2\tan(2x+1) \cot x^2\right]$$

(d) $s(x) = \dfrac{1+\csc x}{1-\csc x}$

solution:

$$s'(x) = \frac{\left[1-\csc x\right]\left[-\csc x \cot x\right] - \left[1+\csc x\right]\left[\csc x \cot x\right]}{(1-\csc x)^2}$$

$$= \frac{\csc x \cot x\left[\csc x - 1 - 1 - \csc x\right]}{(1-\csc x)^2}$$

$$= \csc x \cot x \cdot \frac{-2}{(1-\csc x)^2}$$

$$= \frac{-2\csc x \cot x}{(1-\csc x)^2}$$

(e) $f(x) = x^3 \sin x \cos x$

Solution:

$2 \sin x \cos x = \sin 2x$

$f(x) = \frac{1}{2} x^3 \sin 2x$

$f'(x) = \frac{1}{2} \left[x^3 \cos 2x \cdot 2 + \sin 2x \cdot 3x^2 \right]$

$$\boxed{f'(x) = x^3 \cos 2x + \frac{3}{2} x^2 \sin 2x}$$

(f) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$

solution:

$y^{\frac{1}{2}} = 1 - x^{\frac{1}{2}}$

$y = \left(1 - \sqrt{x}\right)^2$

$y' = 2\left(1 - \sqrt{x}\right) \cdot -\frac{1}{2\sqrt{x}}$

$$y' = \frac{\sqrt{x} - 1}{\sqrt{x}}$$

2 Find $\frac{dy}{dx}$ for the follows

a $y = \cot^2 x$

Solution:

$y' = 2\cot x \cdot -\csc^2 x$

$$\boxed{y' = -2\cot x \csc^2 x}$$

~~b $f(x) = \frac{1}{x}$~~

b $x^2 + y^2 = x$

Solution:

$2x + 2yy' = 1$

$y' = \frac{1-2x}{2y}$

c $y = \frac{\sin x}{1-\cos x}$

solution:

$$y' = \frac{[1-\cos x]\cdot \cos x - \sin x \cdot +\sin x}{(1-\cos x)^2}$$

$$= \frac{\cos x - \cos^2 x - \sin^2 x}{(1-\cos x)^2}$$

$$= \frac{\cos x - [\cos^2 x + \sin^2 x]}{(1-\cos x)^2} = \frac{\cos x - 1}{(1-\cos x)^2} = \frac{\cos x - 1}{(\cos x - 1)^2} = \boxed{\frac{1}{\cos x - 1}}$$

(5)

3 Find the points on the curve

$$y = 2x^3 - 3x^2 - 12x + 20$$

where the tangent is parallel to the x - axis

Solution

tangent line is horizontal

$\bar{y} = m = \text{slope} = 0$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1, 2$

$(-1, f(-1)) = (-1, 27)$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 20$

$= -2 - 3 + 12 + 20$

$= 27$

$(2, f(2)) = (2, 0)$

$f(2) = 2 \cdot (2)^3 - 3(2)^2 - 12 \cdot 2 + 20$

$= 16 - 12 - 24 + 20$

$= 0$

6

(4) For what values of the constant a, if any, is

$$f(x) = \begin{cases} \sin 2x & , x \le 0 \\ ax & , x > 0 \end{cases}$$

(i) Continuous at $x = 0$?

(ii) Differentiable at $x = 0$

Solution

Continuous at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$0 = 0$$

So $f(x)$ is continuous for all a

$$f'(x) = \begin{cases} 2\cos 2x & , x < 0 \\ a & , x > 0 \end{cases}$$

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$$

$$\lim_{x \to 0^-} 2\cos 2x = \lim_{x \to 0^+} a$$

$$\boxed{2 = a}$$

5 Find the normals to the curve $xy+2x-y=0$ that are parallel to the line $2x+y=0$

Solution

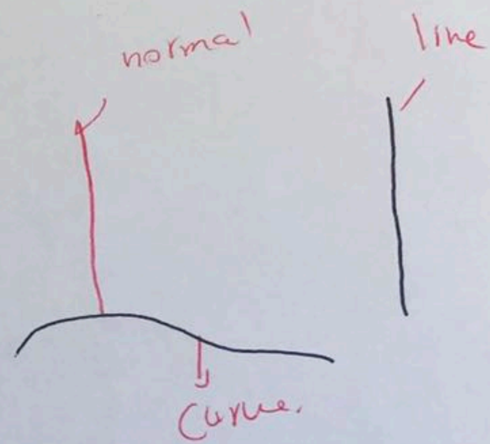
normal to the curve is parallel to line.

so they have same slope.

slope of the line $2x+y=0$ is $\boxed{-2}$

$$\boxed{\text{So slope of the normal is } -2}$$

also slope of the curve is y'



$$xy'+y+2-y'=0$$

$$y'(x-1)=-2-y$$

$$y'=\frac{-2-y}{x-1}=\frac{-(2+y)}{x-1}$$

slope of the curve is $\frac{(-2+y)}{x-1}$

slope of the normal to the curve is $\frac{x-1}{2+y}$

which is -2

$$-2=\frac{x-1}{2+y} \Longrightarrow -4-2y=x-1$$

$$\boxed{x=-3-2y}$$

$xy + 2x - y = 0$

$(-3-2y)y + 2(-3-2y) - y = 0$

$-3y - 2y^2 - 6 - 4y - y = 0$

$-2y^2 - 8y - 6 = 0$

$y^2 + 4y + 3 = 0$

$(y+1)(y+3) = 0$

$y = -1 \quad , \quad y = -3$

$y = -1 \longrightarrow x = -3 - 2(-1) = -3 + 2 = -1 \qquad (-1, -1)$

$y = -3 \longrightarrow x = -3 - 2(-3) = -3 + 6 = 3 \qquad (3, -3)$

two points $(-1,-1)$, $(3,-3)$ slope $= -2$

① $y - -1 = -2(x - -1) \longrightarrow y + 1 = -2x - 2 \longrightarrow \boxed{y = -2x - 3}$

② $y - -3 = -2(x - 3) \longrightarrow y + 3 = -2x + 6 \longrightarrow \boxed{y = -2x + 3}$

(6) Find the linearization of the following functions at the given points

(a) $f(x) = \tan x$, $x = \frac{\pi}{4}$

$L(x) = f'(a)(x-a) + f(a)$

$f(\frac{\pi}{4}) = \tan \frac{\pi}{4} = 1$

$f'(x) = \sec^2 x$, $f'(\frac{\pi}{4}) = \sec^2 \frac{\pi}{4} = 2$

$L(x) = 2(x - \frac{\pi}{4}) + 1$

$\boxed{L(x) = 2x - \frac{\pi}{2} + 1}$

(b) $g(x) = \frac{1}{x}$, $x = 1$

$L(x) = f(a) + f'(a)(x-a)$

$g'(x) = \frac{-1}{x^2}$

$g'(1) = -1$

$g(1) = 1$

$L(x) = 1 - 1(x-1)$

$L(x) = 1 - x + 1$

$\boxed{L(x) = 2 - x}$

(c) $h(x) = \frac{x^2}{x^2+1}$, $x=0$

Solution

$h(0) = 0$

$h'(x) = \frac{(x^2+1)\cdot 2x - x^2 \cdot 2x}{(x^2+1)^2}$

$h'(0) = \frac{0-0}{1} = 0$

so $L(x) = 0 + 0(x-0)$

$\boxed{L(x) = 0}$

(d) $f(x) = 1 + \cos\theta$, $\theta = \frac{\pi}{3}$

Solution

$f'(x) = -\sin\theta$

$f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = \frac{-\sqrt{3}}{2}$

$f\left(\frac{\pi}{3}\right) = 1 + \cos\frac{\pi}{3} = 1 + \frac{1}{2} = \frac{3}{2}$

$L(x) = \frac{3}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$

$L(x) = \frac{3}{2} - \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}\pi}{6} \rightarrow \boxed{L(x) = -\frac{\sqrt{3}}{2}x + \frac{9+\sqrt{3}\pi}{6}}$

7. The radius of the circle is increased from 2 to 2.02 m 12

a) Estimate the resulting change in area.

Solution: $A = \pi r^2$

$\boxed{r = 2 \\ dr = 0.02}$

$dA = 2\pi r\, dr$

$= 2\pi \cdot 2 \cdot (0.02)$

$\boxed{dA = 0.08\pi}$

b) . Express this estimate as a percentage of the circle's original data.

$$\frac{dA}{A} = \frac{0.08\pi}{\pi \cdot 4} = \frac{0.08}{4} = 0.02$$

dA is 2% of A