15.3 Area by Double Integration

* Recall the Riemann sum in the definition of a double integral $S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$.

* If we let $f(x,y) = 1$, then the Riemann sum becomes $S_n = \sum_{k=1}^{n} \Delta A_k$ which is the sum of the areas of small rectangles of R.

* We define the area of a closed, bounded plane region R by:

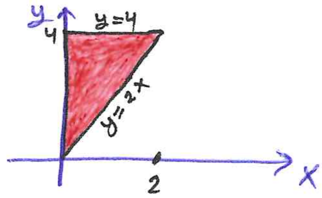
$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA$$

Exp Find the area of the region R bounded by

1. $x = 0$, $y = 2x$ and $y = 4$.

$$A = \int_0^2 \int_{2x}^{4} dy\,dx = \int_0^2 (4-2x)\,dx = 4$$

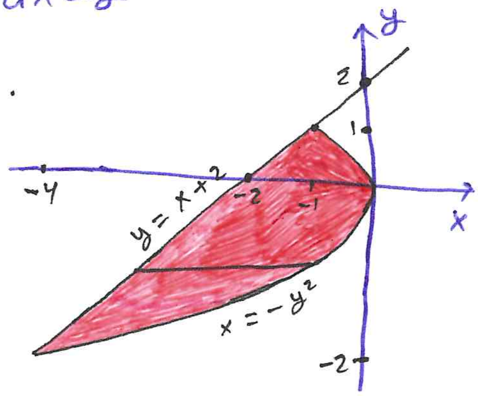
$$= \int_0^4 \int_0^{\frac{y}{2}} dx\,dy$$



2. The parabola $x = -y^2$ and the line $y = x+2$.

$$A = \int_{-2}^{1} \int_{y-2}^{-y^2} dx\,dy = \int_{-2}^{1} (2-y-y^2)\,dy$$

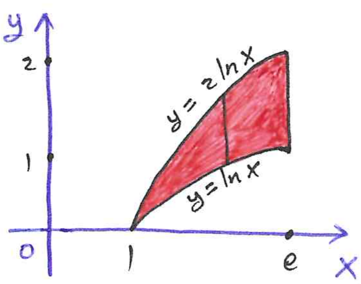
$$= \frac{9}{2}$$



3. The curve $y = \ln x$ and $y = 2\ln x$ and the line $x = e$

$$A = \int_1^e \int_{\ln x}^{2\ln x} dy\,dx = \int_1^e \ln x\,dx$$

$$= (x\ln x - x)\Big|_1^e = 1$$



* Average Value:

- Remember that the average value of an integrable function of one variable, $f(x)$, on closed interval $[a,b]$ is $av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

- We now define the average value of an integrable function f over a region R as follows:

$$av(f) = \frac{1}{A}\iint_R f\,dA,$$ where A is the area of R.

Exp Find the average value of $f(x,y) = \frac{1}{xy}$ over the square $\ln 2 \leq x \leq 2\ln 2$, $\ln 2 \leq y \leq 2\ln 2$.

- $$A = \int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2} dy\,dx = \int_{\ln 2}^{2\ln 2} \ln 2\,dx = (\ln 2)^2$$

- $$av(f) = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2} \frac{dy\,dx}{xy} = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{1}{x}\left[\ln y\Big|_{y=\ln 2}^{y=2\ln 2}\right] dx$$

$$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{\ln 2}{x}\,dx = \frac{1}{\ln 2}\int_{\ln 2}^{\ln 2} \frac{dx}{x}$$

$$= \frac{1}{\ln 2}\left[\ln x\Big|_{\ln 2}^{\ln 2}\right] = \frac{\ln 2}{\ln 2} = 1$$