

Ch. 8.1 Integration by part :-

Evaluate the integral by using integration by part :-

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\boxed{Q_6} \int_1^e x^3 \ln x \, dx$$

$$\begin{array}{l} u = \ln x \quad dv = x^3 \, dx \\ du = \frac{1}{x} \, dx \quad \swarrow \searrow \quad v = \frac{x^4}{4} \end{array}$$

$$I = \frac{x^4}{4} \ln x \Big|_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{1}{x} \, dx$$

$$= \left(\frac{e^4}{4} - \frac{1}{4} \ln 1 \right) - \int_1^e \frac{x^3}{4} \, dx$$

$$= \frac{e^4}{4} - \left[\frac{x^4}{16} \right]_1^e$$

$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \boxed{\frac{3}{16} e^4 + \frac{1}{16}}$$

$$\boxed{Q_{12}} \int \sin^{-1} y \, dy$$

$$\begin{array}{l} u = \sin^{-1} y \quad dv = dy \\ du = \frac{dy}{\sqrt{1-y^2}} \quad \swarrow \searrow \quad v = y \end{array}$$

$$I = y \sin^{-1} y - \int y \cdot \frac{1}{\sqrt{1-y^2}} \, dy$$

$$\begin{array}{l} \text{let } z = 1-y^2 \\ dz = -2y \, dy \end{array}$$

$$= y \sin^{-1} y - \int \frac{y}{\sqrt{z}} \cdot \frac{dz}{-2y}$$

$$= y \sin^{-1} y + \frac{1}{2} \int z^{-1/2} \, dz = \boxed{y \sin^{-1} y + \sqrt{1-y^2} + C}$$

Q₁₆

$$\int p^4 e^{-p} dp$$

$$I = -p^4 e^{-p} - 4p^3 e^{-p} + 12p^2 e^{-p} - 24p e^{-p} - 24 e^{-p} + C$$

Derivative

Integral

p^4	$\xrightarrow{+}$	$-e^{-p}$
$4p^3$	$\xrightarrow{-}$	e^{-p}
$12p^2$	$\xrightarrow{+}$	$-e^{-p}$
$24p$	$\xrightarrow{-}$	e^{-p}
24	$\xrightarrow{+}$	$-e^{-p}$
0		

Q₂₂ $\int e^{-y} \cos y dy$

First time

let $u = \cos y$	$\swarrow$	$dv = e^{-y} dy$
$du = -\sin y dy$	$\xleftarrow{-\int} \xrightarrow{\int}$	$v = -e^{-y}$

$$\int e^{-y} \cos y dy = -\cos y e^{-y} - \int \sin y e^{-y} dy$$

Second time

let $u = \sin y$	$dv = e^{-y} dy$
$du = \cos y dy$	$v = -e^{-y}$

$$\int e^{-y} \cos y dy = -\cos y e^{-y} - (-\sin y e^{-y} + \int e^{-y} \cos y dy)$$

$$\int e^{-y} \cos y dy = \sin y e^{-y} - \cos y e^{-y} - \int e^{-y} \cos y dy$$

$$\int e^{-y} \cos y dy = e^{-y} [\sin y - \cos y] / 2 + C$$

Q30

$$\int Z (\ln Z)^2 dZ$$

By substitution:-

$$\text{let } y = \ln Z \rightarrow Z = e^y$$

$$dZ = e^y dy$$

$$I = \int y^2 e^{2y} dy$$

$$I = \frac{1}{2} y^2 e^{2y} - \frac{1}{2} y e^{2y} + \frac{1}{4} e^{2y} + C$$

$$= \frac{1}{2} (\ln Z)^2 Z^2 - \frac{1}{2} (\ln Z) Z^2 + \frac{1}{4} Z^2 + C$$

$$= \frac{1}{2} Z^2 (\ln Z)^2 - \frac{1}{2} Z^2 (\ln Z) + \frac{1}{4} Z^2 + C$$

Derivative

y^2	+	
$2y$	-	
2	+	
0		

Integral

e^{2y}
$\frac{e^{2y}}{2}$
$\frac{e^{2y}}{4}$
$\frac{e^{2y}}{8}$

Q33

$$\int x (\ln x)^2 dy$$

"Exactly Q30"

Q36

$$\int \frac{(\ln x)^3}{x} dx$$

$$\text{let } y = \ln x$$

$$dy = \frac{1}{x} dx$$

$$\int y^3 dy$$

$$I = \frac{y^4}{4} + C$$

$$I = \frac{\ln^4 x}{4} + C$$

Q39

$$\int x^3 \sqrt{x^2 + 1} \, dx$$

$$\text{let } \boxed{y = x^2 + 1}^*$$

$$\boxed{dy = 2x \, dx}^*$$

$$I = \int x^3 \sqrt{y} \frac{dy}{2x}$$

$$= \frac{1}{2} \int (y-1) \sqrt{y} \, dy$$

$$= \frac{1}{2} \int \left(y^{\frac{3}{2}} - y^{\frac{1}{2}} \right) dy$$

$$= \frac{1}{2} \left[\frac{2}{5} y^{5/2} - \frac{2}{3} y^{3/2} \right] + C$$

$$= \frac{1}{2} \left[\frac{2}{5} (x^2 + 1)^{5/2} - \frac{2}{3} (x^2 + 1)^{3/2} \right] + C$$

$$I = \frac{1}{5} (x^2 + 1)^{5/2} - \frac{1}{3} (x^2 + 1)^{3/2} + C$$

Q46

$$\int \sqrt{x} e^{\sqrt{x}} dx$$

let $y = \sqrt{x} \rightarrow y^2 = x$

$$2y dy = dx$$

$$I = \int y e^y 2y dy$$

$$I = \int 2y^2 e^y dy$$

$$= 2y^2 e^y - 4y e^y + 4 e^y + C$$

$$= 2x e^{\sqrt{x}} - 4\sqrt{x} e^{\sqrt{x}} + 4 e^{\sqrt{x}} + C$$

Derivative	Integral
$2y^2$	e^y
$4y$	e^y
4	e^y
0	e^y

Q49

$$\int_{\frac{2}{\sqrt{3}}}^2 t \sec^{-1} t dt$$

$$\boxed{\begin{array}{l} \text{let } u = \sec^{-1} t \\ du = \frac{1}{t\sqrt{t^2-1}} dt \end{array} \quad \begin{array}{l} dv = t dt \\ v = \frac{t^2}{2} \end{array}}$$

$$\int_{\frac{2}{\sqrt{3}}}^2 t \sec^{-1} t = \left[\frac{t^2 \sec^{-1} t}{2} \right]_{\frac{2}{\sqrt{3}}}^2 - \frac{1}{2} \int_{\frac{2}{\sqrt{3}}}^2 \frac{t}{\sqrt{t^2-1}} dt$$

$$= \frac{1}{2} \left[4 \sec^{-1}(2) - \frac{4}{3} \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) \right] - I_1$$

$$= \frac{1}{2} \left[4 \cos^{-1}\left(\frac{1}{2}\right) - \frac{4}{3} \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \right] - I_1 = \frac{5\pi}{9} - I_1$$

We need to find $I_1 = \frac{1}{2} \int_{2/\sqrt{3}}^2 \frac{t}{\sqrt{t^2-1}} dt$

$$= \frac{1}{2} \int \frac{1}{\sqrt{y}} \frac{dy}{2}$$

$$= \frac{1}{4} \int_{1/3}^3 y^{-1/2} dy$$

$$= \frac{1}{4} \cdot 2 \left[y^{1/2} \right]_{1/3}^3$$

$$= \frac{1}{2} \left[\sqrt{3} - \frac{1}{\sqrt{3}} \right]$$

$$I_1 = \frac{1}{2} \left[\frac{2}{\sqrt{3}} \right] = \frac{1}{\sqrt{3}}$$

By substitution:

$$\text{let } y = t^2 - 1$$

$$dy = 2t dt$$

$$* t = \frac{2}{\sqrt{3}} \rightarrow y = \frac{1}{3}$$

$$t = 2 \rightarrow y = 3$$

Then: $\int_{2/\sqrt{3}}^2 t \sec^{-1} t dt = \frac{5\pi}{9} - \frac{3\sqrt{3}}{3\sqrt{3}}$

$$I = \frac{5\pi - 3\sqrt{3}}{9}$$

#.

8.2 Trigonometric Integrals :-

$$[6] \int \cos^3 4x \, dx$$

$$\int \cos 4x (\cos^2 4x) \, dx$$

$$= \int \cos 4x (1 - \sin^2 4x) \, dx$$

$$\text{Let } u = \sin 4x \\ du = 4 \cos 4x \, dx$$

$$= \frac{1}{4} \int (1 - u^2) \, du$$

$$= \frac{1}{4} \left[u - \frac{u^3}{3} \right] + C$$

$$I = \frac{1}{4} \left[\sin(4x) - \frac{\sin^3(4x)}{3} \right] + C$$

$$= \frac{\sin(4x)}{4} - \frac{\sin^3(4x)}{12} + C$$

$$\boxed{11} \int \sin^3 x \cos^3 x \, dx$$

$$\int \sin^3 x (1 - \sin^2 x) \cos x \, dx$$

$$\text{let } u = \sin x \\ du = \cos x \, dx$$

$$I = \int u^3 (1 - u^2) \, du$$

$$= \int u^3 - u^5 \, du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} + C$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$$

$$\boxed{17} \int_0^{\pi} 8 \sin^4 x \, dx$$

$$\int_0^{\pi} 8 (\sin^2 x)^2 \, dx$$

$$= \int_0^{\pi} 8 \left(\frac{1 - \cos(2x)}{2} \right)^2 \, dx$$

$$= \int_0^{\pi} \frac{8}{4} [1 - 2\cos(2x) + \cos^2(2x)] \, dx$$

$$= 2 \int_0^{\pi} 1 - 2\cos(2x) + \left(\frac{1 + \cos(4x)}{2} \right) \, dx$$

$$\begin{aligned}
 I &= 2 \left[x - \frac{2 \sin(2x)}{2} + \frac{1}{2} x + \frac{1}{2} \frac{\sin(4x)}{4} \right]_0^\pi \\
 &= 2 \left[\frac{3x}{2} - \sin(2x) + \frac{1}{8} \sin(4x) \right]_0^\pi \\
 &= 3\pi
 \end{aligned}$$

[22] $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta$

$$\int_0^{\pi/2} \sin^2 2\theta \cos 2\theta (1 - \sin^2 2\theta) d\theta$$

Let $u = \sin 2\theta$
 $du = 2 \cos 2\theta d\theta$

$$I = \int_0^0 u^2 (1 - u^2) \frac{du}{2}$$

* $\theta = 0 \rightarrow u = 0$

$\theta = \frac{\pi}{2} \rightarrow u = 0$

= Zero

[28] $\int_0^{\pi/6} \sqrt{1 + \sin x} dx$ "Eliminating Square root"

$$\int_0^{\pi/6} \frac{\sqrt{1 - \sin^2 x}}{\sqrt{1 - \sin x}} dx = \int_0^{\pi/6} \frac{|\cos x|}{\sqrt{1 - \sin x}} dx$$

$$I = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}} dx$$

$$= \int_1^{1/2} \frac{-1}{\sqrt{u}} du$$

$$= \int_{1/2}^1 \frac{1}{\sqrt{u}} du$$

$$= 2 u^{1/2} \Big|_{1/2}^1$$

$$= 2 \left[1 - \sqrt{\frac{1}{2}} \right] = 2 \left[\frac{\sqrt{2}-1}{\sqrt{2}} \right]$$

$$= 2 - \sqrt{2}$$

$$\text{let } u = 1 - \sin x$$

$$du = -\cos x dx$$

$$* x=0 \rightarrow u=1$$

$$x = \frac{\pi}{6} \rightarrow u = \frac{1}{2}$$

$$\boxed{34} \int \sec x \tan^2 x \, dx$$

$$\int \sec x (\sec^2 x - 1) \, dx$$

$$\int \sec^3 x \, dx - \int \sec x \, dx$$

By part By ln

let $U = \sec x$ $dU = \sec x \tan x$	$\xleftrightarrow{-\int}$	$dV = \sec^2 x \, dx$ $V = \tan x$
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$$\int \sec^3 x \, dx - \int \sec x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx - \int \sec x \, dx$$

$$\underline{\int \sec x \tan^2 x \, dx} = \sec x \tan x - \underline{\int \tan^2 x \sec x \, dx} - \int \sec x \, dx$$

$$2 \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x \, dx$$

$$= \sec x \tan x - \ln |\sec x + \tan x| + C$$

$$\text{Then } \int \sec x \tan^2 x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$$

$$\boxed{41} \int \sec^4 x \, dx$$

$$\int \sec^2 x \sec^2 x \, dx$$

$$\int (1 + \tan^2 x) \sec^2 x \, dx$$

$$\text{let } u = \tan x \\ du = \sec^2 x \, dx$$

$$\int (1 + u^2) \, du$$

$$I = u + \frac{u^3}{3} + C$$

$$I = \tan x + \frac{1}{3} \tan^3 x + C$$

$$\boxed{52} \int \sin(2x) \cos(3x) \, dx$$

Notes

$$\sin(mx) \cos(nx) =$$

$$= \frac{1}{2} [\sin(n-m)x + \sin(n+m)x]$$

$$= \frac{1}{2} \int \sin(-x) + \sin(5x) \, dx$$

$$= \frac{1}{2} \int -\sin x \, dx + \frac{1}{2} \int \sin(5x) \, dx$$

$$= \frac{1}{2} \cos x + \frac{1}{2} \frac{\cos(5x)}{5} + C$$

$$= \frac{1}{2} \cos x + \frac{1}{10} \cos(5x) + C$$

$$\boxed{64} \int \frac{\sin^3 x}{\cos^4 x} dx$$

$$\int \frac{\sin^2 x}{\cos^3 x \cos x} dx$$

$$= \int \tan^2 x \sec x dx$$

$$= \int \tan x \tan^2 x \sec x dx$$

$$= \int (\sec^2 x - 1) \tan x \sec x dx$$

$$= \int (u^2 - 1) du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{1}{3} \sec^3 x - \sec x + C$$

$$\text{let } u = \sec x \\ du = \sec x \tan x dx$$