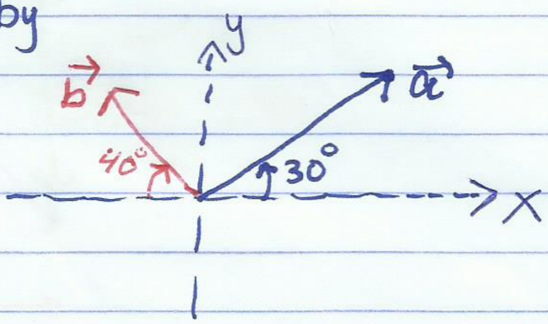
Chapter 3 Vectors

Physical quantities :-

① Scalar quantities : determined by magnitude only
mass , speed , density , work , ...
kg , m/s , kg/m³ , Joul , ...

② Vectors : are determined by magnitude and direction
Position , Displacement , Force , velocity , ...

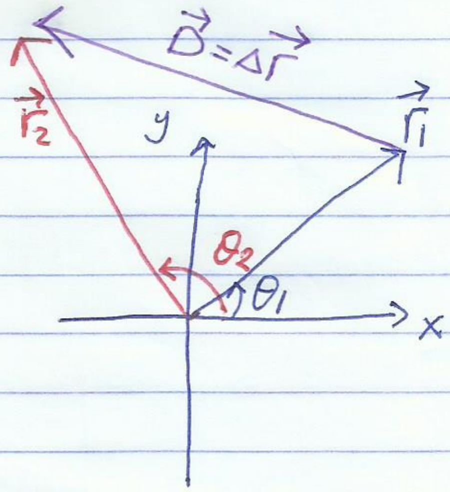
Vectors are represented by an arrow
$\vec{a}$ → magnitude
→ direction



For example: ① Position :

$\vec{a} = 500m$, 30° with +x Counterclock wise
$\vec{b} = 300$ m, 40° with −x clock wise

② Displacement = change of Position
$\vec{D} = \Delta\vec{r}$
$\vec{D} = \vec{r_2} - \vec{r_1}$

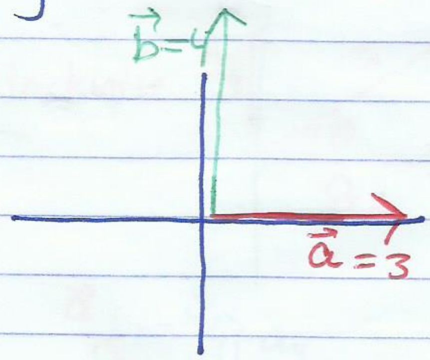


example: At t_1, moving particle at $\vec{r_1}$
At t_2, moving Particle at $\vec{r_2}$
as shown

$\vec{D} = \vec{r_2} - \vec{r_1} \Longrightarrow \vec{r_1} + \vec{D} = \vec{r_2}$

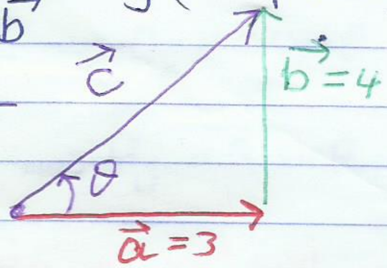
the law of adding vectors
(Vector sum
Resultant)

example: you have the following 2 vectors



Find: Geometrically (Graphically)

① $\vec{c} = \vec{a} + \vec{b}$

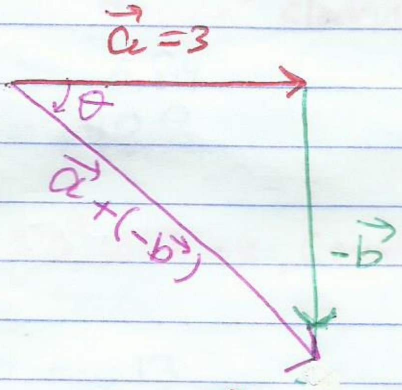


$c = \sqrt{a^2+b^2}$

$= 5$

② $\vec{D} = \vec{a} - \vec{b}$

$\vec{D} = \vec{a} + (-\vec{b})$



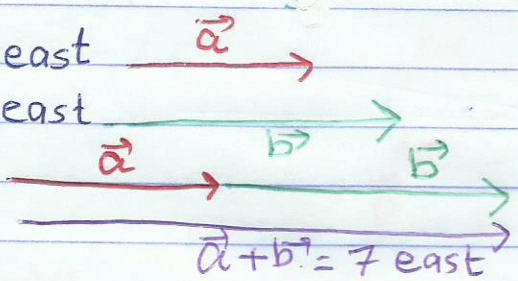
$D = \sqrt{a^2+b^2}$

$= 5$

Checkpoint 1:- $\vec{a} = 3$, east

$\vec{b} = 4$; east

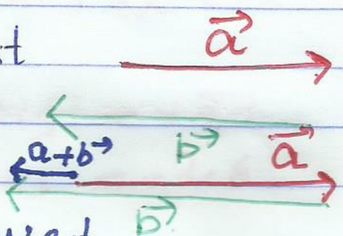
$\vec{a} + \vec{b}$



② $\vec{a} = 3$; east

$\vec{b} = 4$; West

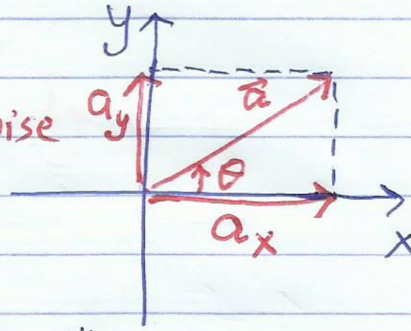
$\vec{a} + \vec{b} =$



$\vec{a} + \vec{b} = -1$, west

Components of Vectors:-

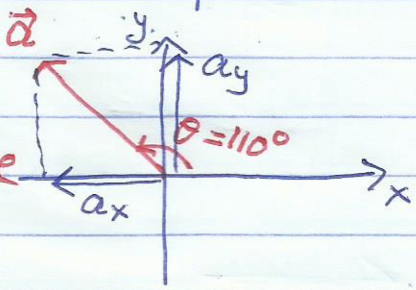
$\vec{a} = a$, at θ with $+x$, Counter clockwise



$\cos\theta = \frac{a_x}{a} \Rightarrow a_x = a\cos\theta$

$\sin\theta = \frac{a_y}{a} \Rightarrow a_y = a\sin\theta$

example:- $\vec{a} = 20$; at $110°$ with $+x$ counter clockwise



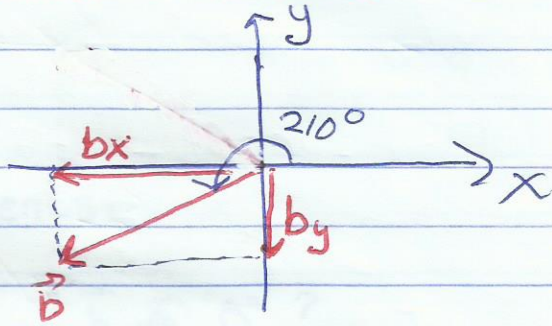
$a_x = 20\cos 110 = -6.84$

$a_y = 20\sin 110 = +18.8$

$\vec{b} = 15$; at $210°$ with $+x$, Counter Clockwise

$b_x = 15\cos 210 = -13$

$b_y = 15\sin 210 = -7.5$

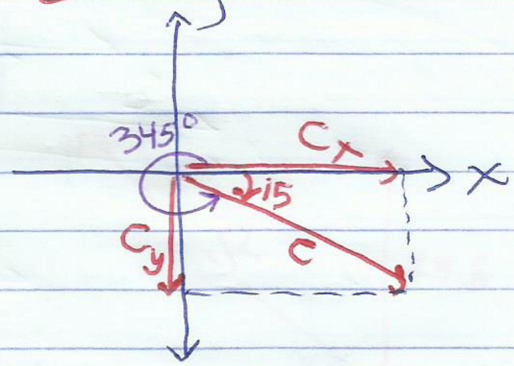


$\vec{C} = 10$, at $15°$ with $+x$ clockwise

$C_x = C\cos 345$

$C_x = 10\cos 345 = +9.7$

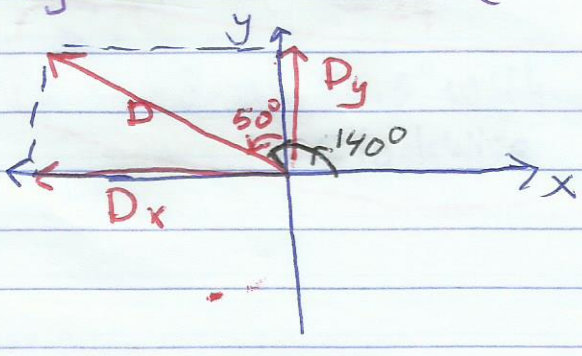
$C_y = 10\sin 345 = -2.6$



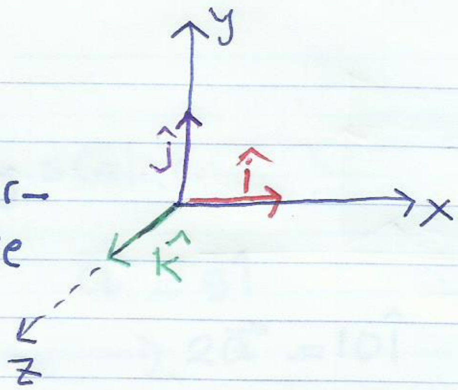
$\vec{D} = 25$, at $50°$, with $+y$ Counter Clockwise

$D_x = 25\cos 140$
$= -19$

$D_y = 25\sin 140$
$= +16.1$



Unit vectors:



$\vec{a} = 20$, at $^{+}110°$ with $+x$ Counter-clockwise

$\vec{a} = -6.84\hat{i} + 18.8\hat{j}$

$\vec{b} = 15$, at $^{+}210°$ with $+x$

$\vec{b} = -13\hat{i} - 7.5\hat{j}$

$\vec{C} = 10$, at $^{+}15°$ with $+x$

$\vec{C} = 9.7\hat{i} - 2.6\hat{j}$

$\vec{D} = 25$, at $^{+}50°$, with $+y$

$\vec{D} = -19\hat{i} + 16.1\hat{j}$

ADDing Vectors by Components:

Find the sum of Four vectors $\vec{a}, \vec{b}, \vec{C}, \vec{D}$?

$\vec{R} = \vec{a} + \vec{b} + \vec{C} + \vec{D}$? Sum of Vectors ≡ Resultant

$R_x = a_x + b_x + C_x + D_x$

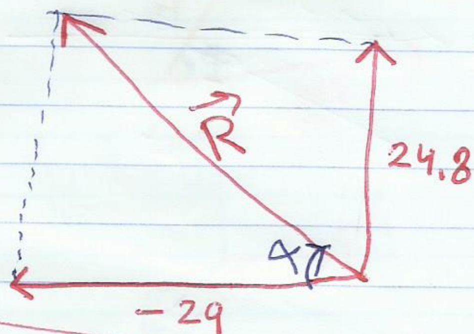
$= -6.84 + {}^{-}13 + 9.7 + {}^{-}19$

$= -29$

$R_y = a_y + b_y + C_y + D_y$

$= 18.8 + {}^{-}7.5 + {}^{-}2.6 + 16.1$

$= 24.8$



$\vec{R} = -29\hat{i} + 24.8\hat{j}$ → $\vec{R} = 38$, at $40.5°$ with $-x$ clockwise

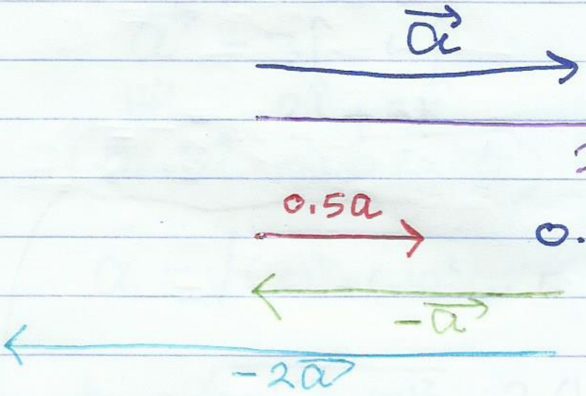
$R = \sqrt{(-29)^2 + (24.8)^2}$

$= 38$

$\alpha = \tan^{-1}\left(\frac{24.8}{-29}\right) = -40.5°$

Multiplying Vectors:

1) Multiplying a Vector by scalar:



$\vec{a} = 5\hat{i}$

$2\vec{a} = 10\hat{i}$

$0.5\vec{a} = 2.5\hat{i}$

$-\vec{a} = -5\hat{i}$

$-2\vec{a} = -10\hat{i}$

$\vec{b} = 3\hat{i} + 4\hat{j} + 6\hat{k}$

$2\vec{b} = 6\hat{i} + 8\hat{j} + 12\hat{k}$

$-3\vec{b} = -9\hat{i} - 12\hat{j} - 18\hat{k}$

2) Multiplying a Vector by a Vector

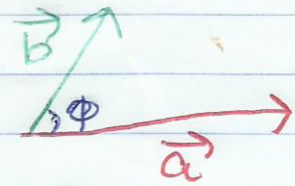
① Scalar product (Dot product)

② Vector product (cross product)

The Scalar Product (The Dot Product)

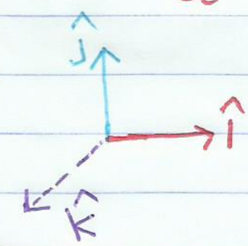
$\vec{a} \cdot \vec{b} = ab\cos\phi$

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = ab\cos\phi$



$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0$

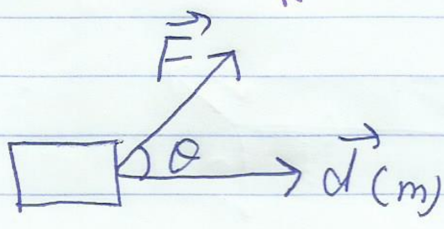


example

$W = \vec{F} \cdot \vec{d}$

$= Fd\cos\theta$

$= Fd\cos\theta \quad N.m = J$



$\vec{a} \cdot \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$

$= a_x b_x + a_y b_y + a_z b_z$

Sample Problem 3.05

What is the angle ϕ between the 2 vectors ?

$$\vec{a} = 3\hat{i} - 4\hat{j}$$

$$\vec{b} = -2\hat{i} + 3\hat{k}$$

$$\vec{a} \cdot \vec{b} = ab\cos\phi$$

$$a = \sqrt{(3)^2 + (-4)^2} = 5$$

$$b = \sqrt{(-2)^2 + (3)^2} = 3.61$$

$$\vec{a} \cdot \vec{b} = (3\hat{i} - 4\hat{j}) \cdot (-2\hat{i} + 3\hat{k}) =$$

$$=$$

$$= -6(\hat{i} \cdot \hat{i}) + 9(\hat{i} \cdot \hat{k}) + 8(\hat{j} \cdot \hat{i}) - 12(\hat{j} \cdot \hat{k})$$

$$= -6 + 0 + 0 + 0$$

$$= -6$$

$$\cos\phi = \frac{\vec{a} \cdot \vec{b}}{ab} = \frac{-6}{(5)(3.61)} = -\frac{6}{18.05} = -0.3324$$

$$\phi = \cos^{-1}(-0.3324) = 109.4^\circ$$

The Vector product (cross Product).

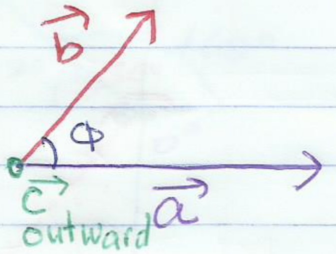
$\vec{a} \times \vec{b} = \vec{c}$

$c = |\vec{a} \times \vec{b}| = ab\sin\phi$

$c = ab\sin\phi$

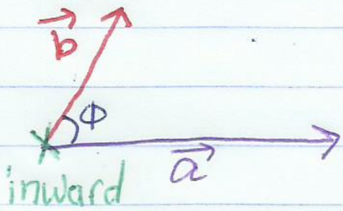
$\vec{c}$ is perpedicular to both $\vec{a}$ + $\vec{b}$

$\vec{c} \perp \vec{a}$ + $\vec{c} \perp \vec{b}$



The direction determined by right-hand rule.

$$\vec{a} \times \vec{b} = (-)\, \vec{b} \times \vec{a}$$

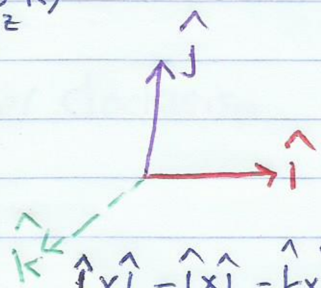


if

$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$



$a_x b_x(\hat{i} \times \hat{i}) + a_x b_y(\hat{i} \times \hat{j}) + a_x b_z(\hat{i} \times \hat{k})$

$0 \quad + a_x b_y \hat{k} \quad + a_x b_z(-\hat{j})$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$,,, $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$,,, $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$,,, $\hat{i} \times \hat{k} = -\hat{j}$

The same for

+

$a_y b_x(\hat{j} \times \hat{i}) + a_y b_y(\hat{j} \times \hat{j}) + a_y b_z(\hat{j} \times \hat{k})$

$a_y b_x(-\hat{k}) \quad + \quad 0 \quad + a_y b_z(\hat{i})$

+

The same for

$a_z b_x(\hat{k} \times \hat{i}) + a_z b_y(\hat{k} \times \hat{j}) + a_z b_z(\hat{k} \times \hat{k})$

$a_z b_x(\hat{j}) \quad + a_z b_y(-\hat{i}) \quad + \quad 0$

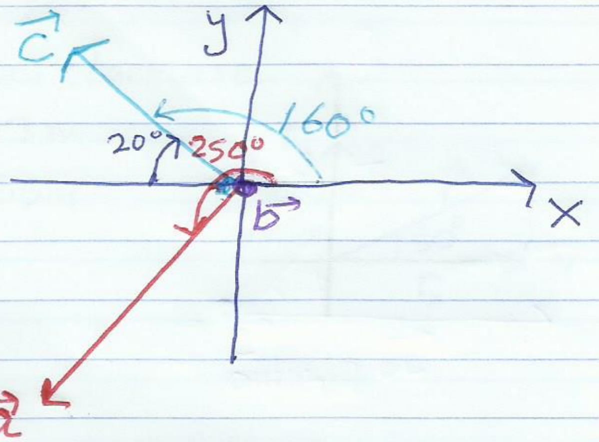
ADD the three lines to get

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}$$

or Do the Determinant (اختياري)

Sample Problem 3.06:-

$\vec{a} = 18$; $250°$ with $+x$ counter clockwise



$\vec{b} = 12$; at $+z$

Find $\vec{c} = \vec{a} \times \vec{b}$

$|\vec{c}| = ab\sin\phi$

$= (18)(12)\sin 90$

$|\vec{c}| = 216$

$\vec{c} \perp \vec{a}$ + $\vec{b}$

$\vec{c}$ in xy plane tobe $\perp \vec{b}$

$\vec{c} \perp \vec{a}$ both in the same plane

the angle between $\vec{c}$ + $-x$ is $20°$ clockwise

or

$\vec{c} = 216$, at $160°$ with $+x$, Counter clockwise

Sample Problem 3.07:

If $\vec{a} = 3\hat{i} - 4\hat{j}$,,, $\vec{b} = -2\hat{i} + 3\hat{k}$

What is $\vec{c} = \vec{a} \times \vec{b}$?

$\vec{c} = (3\hat{i} - 4\hat{j}) \times (-2\hat{i} + 3\hat{k})$

$= 3\hat{i} \times (-2\hat{i}) + (3\hat{i}) \times (3\hat{k}) + (-4\hat{j}) \times (-2\hat{i}) + (-4\hat{j}) \times (3\hat{k})$

$= 0 + 9(-\hat{j}) + 8(-\hat{k}) - 12(\hat{i})$

$\vec{c} = -12\hat{i} - 9\hat{j} - 8\hat{k}$

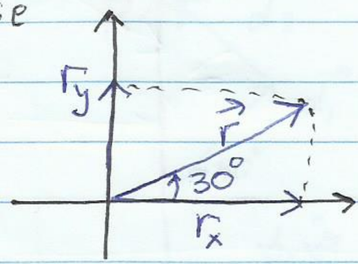
Problems chapter (3):

(3-2) $\vec{r} = 15\ m$; 30° with $+x$ Counterclockwise

$r_x = r\cos 30 = 15\cos 30^\circ = 13\ m$

$r_y = r\sin 30 = 15\sin 30 = 7.5\ m$

$\vec{r} = 13\hat{i} + 7.5\hat{j}\ m$



(3-9) $\vec{a} = 4\hat{i} - 3\hat{j} + 1\hat{k}\ m$

$\vec{b} = -1\hat{i} + 1\hat{j} + 4\hat{k}\ m$

a) Find $\vec{a} + \vec{b}$?

$\vec{r} = \vec{a} + \vec{b} = (4-1)\hat{i} + (-3+1)\hat{j} + (1+4)\hat{k}$

$\vec{r} = 3\hat{i} - 2\hat{j} + 5\hat{k}\ m$

b) Find $\vec{a} - \vec{b}$?

$\vec{d} = \vec{a} - \vec{b} = (4\hat{i} - 3\hat{j} + 1\hat{k}) - (-1\hat{i} + 1\hat{j} + 4\hat{k})$

$\vec{d} = 5\hat{i} - 4\hat{j} - 3\hat{k}\ m$

c) Find a third vector $\vec{c}$ such that $\vec{a} - \vec{b} + \vec{c} = 0$

$\vec{c} = \vec{b} - \vec{a}$

$= (-1\hat{i} + 1\hat{j} + 4\hat{k}) - (4\hat{i} - 3\hat{j} + 1\hat{k})$

$\vec{c} = -5\hat{i} + 4\hat{j} + 3\hat{k}$

(3-11) $\vec{a} = 4\hat{i} + 3\hat{j}\ m$,,, $\vec{b} = -13\hat{i} + 7\hat{j}\ m$

a) Find $\vec{a} + \vec{b}$?

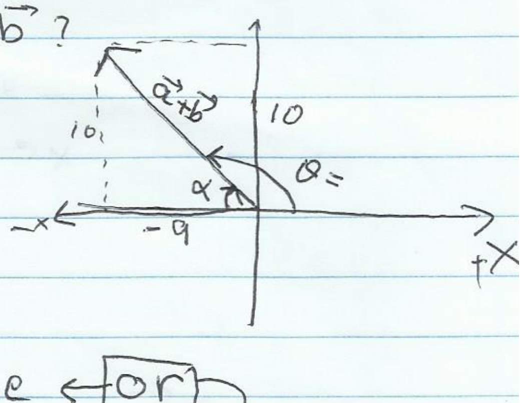
$\vec{r} = \vec{a} + \vec{b} = -9\hat{i} + 10\hat{j}\ m$

b) What is the magnitude of $\vec{a} + \vec{b}$?

$|\vec{a} + \vec{b}| = \sqrt{(-9)^2 + (10)^2} = \sqrt{181}\ m = 13\ m$

c) What is the direction of $\vec{a} + \vec{b}$?

$\vec{a} + \vec{b} = -9\hat{i} + 10\hat{j}$



$\tan\alpha = \frac{10}{9} = 1.11111$

$\alpha = \tan^{-1} 1.111111 = 48^\circ$

$\vec{a} + \vec{b} = 13\ m$; at 48° with $-x$ clockwise $\boxed{\text{or}}$

$= 13\ m$; at 132° with $+x$ Counter clockwise

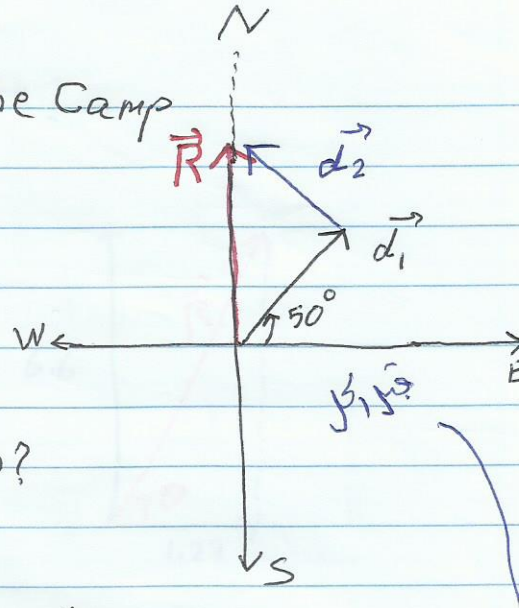
(3-20) $\vec{R} = 5.6\hat{j}$ km, the position of the Camp due snow covering the Area

he travels to...

$\vec{d_1} = 7.8$ km ; 50° North of due East

$\vec{d_1} = 7.8\cos 50\,\hat{i} + 7.8\sin 50\,\hat{j}$

$\vec{d_1} = 5.01\hat{i} + 5.98\hat{j}$ km



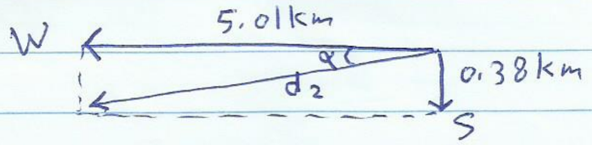
Find $\vec{d_2}$ must he travel to reach camp?

From the graph..

$\vec{d_1} + \vec{d_2} = \vec{R}$

$\vec{d_2} = \vec{R} - \vec{d_1} = 5.6\hat{j} - (5.01\hat{i} + 5.98\hat{j})$

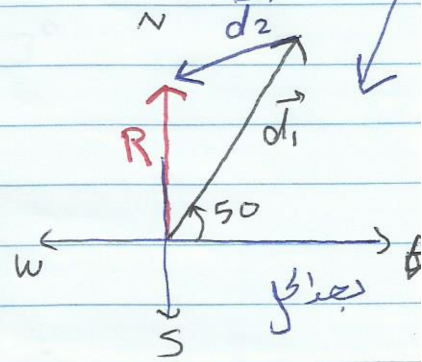
$\vec{d_2} = -5.01\hat{i} - 0.38\hat{j}$ km



$d_2 = \sqrt{(5.01)^2 + (0.38)^2}$

$d_2 = 5.02$ km , $\tan\alpha = \frac{0.38}{5.01}$

$\alpha = \tan^{-1}\left(\frac{0.38}{5.01}\right) = 4.34^\circ$



$\vec{d_2} = 5.02$ km, 4.34° South of due West

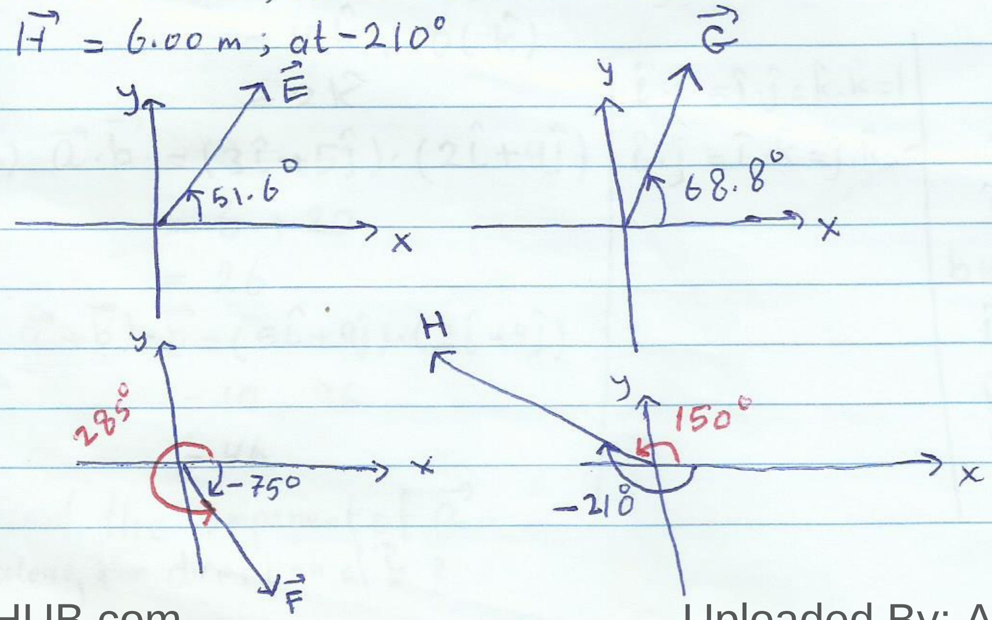
(3-22) You have the following 4 Vectors

$\vec{E} = 6.00$ m ; at $+0.900$ rad $= 6.00$ m, at $+51.6^\circ$

$\vec{G} = 4.00$ m ; at $+1.20$ rad $= 4.00$ m ; at $+68.8^\circ$

$\vec{F} = 5.00$ m ; at -75.0°

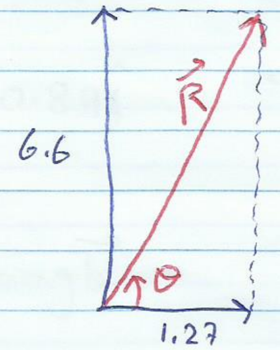
$\vec{H} = 6.00$ m ; at -210°



a) Find the sum of the Four vectors?

$$\vec{R} = \vec{E} + \vec{G} + \vec{F} + \vec{H}$$

$$= (6\cos 51.6\,\hat{i} + 6\sin 51.6\,\hat{j}) +$$
$$+ (4\cos 68.8\,\hat{i} + 4\sin 68.8\,\hat{j}) +$$
$$+ (5\cos 285\,\hat{i} + 5\sin 285\,\hat{j}) +$$
$$+ (6\cos 150\,\hat{i} + 6\sin 150\,\hat{j})$$



$$\vec{R} = 1.27\hat{i} + 6.6\hat{j}\ \text{m}$$

b) Find the magnitude of $\vec{R}$?

$$R = \sqrt{(1.27)^2 + (6.6)^2} = 6.72\,\text{m}$$

c) Find θ?

$$\theta = \tan^{-1}\left(\frac{6.6}{1.27}\right) = 79.1^\circ$$

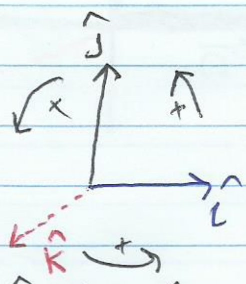
d) Find θ in rad? , $1\,\text{rad} = 57.3^\circ$

$$\theta = \frac{79.1}{57.3} = 1.38\,\text{rad}$$

$$\vec{R} = 6.72\,\text{m},\ \text{at } +79.1^\circ$$
$$\vec{R} = 6.72\,\text{m},\ \text{at } +1.38\,\text{rad}$$

(3-34) $\vec{a} = 3\hat{i} + 5\hat{j}$

$\vec{b} = 2\hat{i} + 4\hat{j}$



find: a) $\vec{a} \times \vec{b} = (3\hat{i} + 5\hat{j}) \times (2\hat{i} + 4\hat{j})$, remember

$$= 12\hat{k} + 10(-\hat{k})$$
$$= 2\hat{k}$$

b) $\vec{a} \cdot \vec{b} = (3\hat{i} + 5\hat{j}) \cdot (2\hat{i} + 4\hat{j})$

$$= 6 + 20$$
$$= 26$$

c) $(\vec{a} + \vec{b}) \cdot \vec{b} = (5\hat{i} + 9\hat{j}) \cdot (2\hat{i} + 4\hat{j})$

$$= 10 + 36$$
$$= 46$$

d) Find the component of $\vec{a}$ along the direction of $\vec{b}$?

$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

but:

$\hat{i} \times \hat{k} = -\hat{j}$

$\hat{k} \times \hat{j} = -\hat{i}$

$\hat{j} \times \hat{i} = -\hat{k}$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = 0$

d) $\vec{a}\cdot\vec{b} = ab\cos\phi$, $a\cos\phi$ = the component of $\vec{a}$ along $\vec{b}$

$b\cos\phi$ = the component of $\vec{b}$ along $\vec{a}$

$\hat{b} = \frac{\vec{b}}{b}$, $\hat{b} \equiv$ is a unit vector ~~at~~ in the direction of $\vec{b}$.

$$\hat{b} = \frac{2\hat{i}+4\hat{j}}{\sqrt{2^2+4^2}} = \frac{2\hat{i}+4\hat{j}}{4.47} = 0.45\hat{i}+0.89\hat{j}$$

$\vec{a}\cdot\hat{b} \equiv$ the component of $\vec{a}$ along $\vec{b}$.

$$\vec{a}\cdot\hat{b} = (3\hat{i}+5\hat{j})\cdot(0.45\hat{i}+0.89\hat{j})$$

$$= 1.35 + 4.45$$

$\vec{a}\cdot\hat{b} = 5.8$; The component of $\vec{a}$ along $\vec{b}$

(3-33) 3 vectors are shown in the Figure

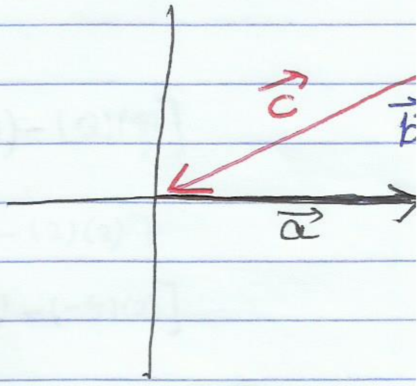
$a=4$, $b=3$, $c=5$

From the Figure $\vec{a}+\vec{b}+\vec{c}=0$

$\vec{c} = -\vec{a}-\vec{b}$, $\vec{a} = 4\hat{i}$

$\vec{b} = 3\hat{j}$

$\vec{c} = -4\hat{i}-3\hat{j}$

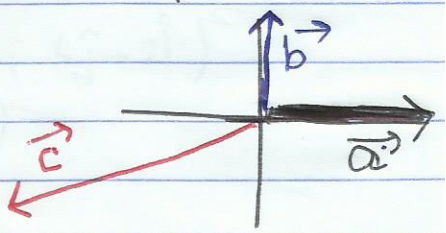


a) Find $\vec{a}\times\vec{b} = 4\hat{i}\times 3\hat{j} = 12\hat{k}$

$\vec{a}\cdot\vec{b} = 4\hat{i}\cdot 3\hat{j} = 0$ extra

$(\vec{a}+\vec{b})\cdot\vec{b} = (4\hat{i}+3\hat{j})\cdot 3\hat{j}$

$= 9$



b) $\vec{a}\times\vec{c} = (4\hat{i})\times(-4\hat{i}-3\hat{j}) = -[4\hat{i}\times(4\hat{i}+3\hat{j})]$

$= (-)[12\hat{k}]$

$= -12\hat{k}$

c) $\vec{b}\times\vec{c} = 3\hat{j}\times(-4\hat{i}-3\hat{j}) = -[3\hat{j}\times(4\hat{i}+3\hat{j})]$

$= -[-12\hat{k}] = +12\hat{k}$

(3-54) $\vec{a}\cdot\vec{b} = 4\hat{i}\cdot 3\hat{j} = 0$

$\vec{a}\cdot\vec{c} = 4\hat{i}\cdot(-4\hat{i}-3\hat{j}) = -16$

$\vec{b}\cdot\vec{c} = 3\hat{j}\cdot(-4\hat{i}-3\hat{j}) = -9$

(3 – 63)

$\vec{d_1} = -3\hat{i} + 3\hat{j} + 2\hat{k}$ m
$\vec{d_2} = -2\hat{i} - 4\hat{j} + 2\hat{k}$ m
$\vec{d_3} = 2\hat{i} + 3\hat{j} + 1\hat{k}$ m

what are:

a) $\vec{d_1} \cdot (\vec{d_2} + \vec{d_3}) = (-3\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (0\hat{i} - 1\hat{j} + 3\hat{k})$
$= 0 + (3)(-1) + 2(3)$
$= -3 + 6$
$= +3$

b) $\vec{d_1} \cdot (\vec{d_2} \times \vec{d_3}) = ?$

$$\vec{d_2} \times \vec{d_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -4 & +2 \\ +2 & +3 & +1 \end{vmatrix} = \hat{i}\left[-4(1) - (2)(3)\right] - \hat{j}\left[-2(1) - (2)(2)\right] + \hat{k}\left[-2(3) - (-4)(2)\right]$$

$= -10\hat{i} + 6\hat{j} + 2\hat{k}$

$\vec{d_1} \cdot (\vec{d_2} \times \vec{d_3}) = (-3\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (-10\hat{i} + 6\hat{j} + 2\hat{k})$
$= (-3)(-10) + (3)(6) + (2)(2)$
$= 30 + 18 + 4$
$= 52$

c) $\vec{d_1} \times (\vec{d_2} + \vec{d_3})$?

$\vec{d_2} + \vec{d_3} = -1\hat{j} + 3\hat{k}$

$$\vec{d_1} \times (\vec{d_2} + \vec{d_3}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & +3 & +2 \\ 0 & -1 & +3 \end{vmatrix} = \hat{i}\left[(3)(3) - (2)(-1)\right] - \hat{j}\left[(-3)(3) - (2)(0)\right] + \hat{k}\left[(-3)(-1) - (3)(0)\right]$$

$= 11\hat{i} + 9\hat{j} + 3\hat{k}$