Boolean Algebra and Logic Gates

ENCS2340 - Digital Systems

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Presentation Outline

- **Boolean Algebra**
- **Boolean Functions and Truth Tables**
- **DeMorgan's Theorem**
- **Algebraic manipulation and expression simplification**
- Logic gates and logic diagrams
- Minterms and Maxterms
- Sum-Of-Products and Product-Of-Sums
- Additional Gates and Symbols

Boolean Algebra

- Introduced by George Boole in 1854
- Two-valued Boolean algebra is also called switching algebra
- A set of two values: $B = \{0, 1\}$
- Three basic operations: **AND**, **OR**, and **NOT**
- The **AND** operator is denoted by a dot (•)
  - $x \cdot y$ or xy is read: x **AND** y
- The **OR** operator is denoted by a plus (**+**)
  - $x + y$ is read: x **OR** y
- The **NOT** operator is denoted by (**'**) or an overbar ($\overline{\ }$).
  - x' or $\overline{x}$ is the complement of x

AND, OR, and NOT Operators

- The following tables define $x \cdot y$, $x + y$, and x'
- $x \cdot y$ is the **AND** operator
- $x + y$ is the **OR** operator
- x' is the **NOT** operator

x y	x·y
0 0	0
0 1	0
1 0	0
1 1	1

x y	x+y
0 0	0
0 1	1
1 0	1
1 1	1

x	x'
0	1
1	0

Postulates of Boolean Algebra

1. Closure: the result of any Boolean operation is in $B = \{0, 1\}$
2. Identity element with respect to + is 0: $x + 0 = 0 + x = x$

   Identity element with respect to $\cdot$ is 1: $x \cdot 1 = 1 \cdot x = x$
3. Commutative with respect to +: $x + y = y + x$

   Commutative with respect to $\cdot$: $x \cdot y = y \cdot x$
4. $\cdot$ is distributive over +: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

   + is distributive over $\cdot$: $x + (y \cdot z) = (x + y) \cdot (x + z)$
5. For every x in B, there exists x' in B (called complement of x) such that: $x + x' = 1$ and $x \cdot x' = 0$

Boolean Functions

- Boolean functions are described by expressions that consist of:
  - Boolean variables, such as: x, y, etc.
  - Boolean constants: 0 and 1
  - Boolean operators: AND ($\cdot$), OR (+), NOT (')
  - Parentheses, which can be nested
- Example: $f = x(y + w'z)$
  - The dot operator is implicit and need not be written
- Operator precedence: to avoid ambiguity in expressions
  - Expressions within parentheses should be evaluated first
  - The NOT (') operator should be evaluated second
  - The AND ($\cdot$) operator should be evaluated third
  - The OR (+) operator should be evaluated last

Truth Table

- A truth table can represent a Boolean function
- List all possible combinations of 0's and 1's assigned to variables
- If *n* variables then 2^n rows
- Example: Truth table for $f = xy' + x'z$

x	y	z	y'	xy'	x'	x'z	f = xy'+ x'z
0	0	0	1	0	1	0	0
0	0	1	1	0	1	1	1
0	1	0	0	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	0	1
1	0	1	1	1	0	0	1
1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0

DeMorgan's Theorem

- $(x + y)' = x'\,y'$
- $(x\,y)' = x' + y'$

Can be verified
Using a Truth Table

x	y	x'	y'	x+y	(x+y)'	x'y'	x y	(x y)'	x'+ y'
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

Identical **Identical**

- Generalized DeMorgan's Theorem:
- $(x_1 + x_2 + \cdots + x_n)' = x_1' \cdot x_2' \cdot \cdots \cdot x_n'$
- $(x_1 \cdot x_2 \cdot \cdots \cdot x_n)' = x_1' + x_2' + \cdots + x_n'$

Complementing Boolean Functions

- What is the complement of $f = x'yz' + xy'z'$?
- Use DeMorgan's Theorem:
  - Complement each variable and constant
  - Interchange AND and OR operators
- So, what is the complement of $f = x'yz' + xy'z'$?

  Answer: $f' = (x + y' + z)(x' + y + z)$

- **Example 2:** Complement $g = (a' + bc)d' + e$
- **Answer:** $g' = (a(b' + c') + d)e'$

Algebraic Manipulation of Expressions

- The objective is to acquire skills in manipulating Boolean expressions, to transform them into simpler form.

- **Example 1:** prove $x + xy = x$ (absorption theorem)

- **Proof:** $x + xy = x \cdot 1 + xy$ $x \cdot 1 = x$

  $= x \cdot (1 + y)$ Distributive $\cdot$ over +

  $= x \cdot 1 = x$ $(1 + y) = 1$

- **Example 2:** prove $x + x'y = x + y$ (simplification theorem)

- **Proof:** $x + x'y = (x + x')(x + y)$ Distributive + over $\cdot$

  $= 1 \cdot (x + y)$ $(x + x') = 1$

  $= x + y$

Consensus Theorem

- **Prove that:** $xy + x'z + yz = xy + x'z$ (consensus theorem)
- **Proof:** $xy + x'z + yz$

$= xy + x'z + 1 \cdot yz$	$yz = 1 \cdot yz$
$= xy + x'z + (x + x')yz$	$1 = (x + x')$
$= xy + x'z + xyz + x'yz$	Distributive $\cdot$ over +
$= xy + xyz + x'z + x'yz$	Associative commutative +
$= xy \cdot 1 + xyz + x'z \cdot 1 + x'zy$	$xy = xy \cdot 1, \;\; x'yz = x'zy$
$= xy(1 + z) + x'z(1 + y)$	Distributive $\cdot$ over +
$= xy \cdot 1 + x'z \cdot 1$	$1 + z = 1, \;\; 1 + y = 1$
$= xy + x'z$	$xy \cdot 1 = xy, \;\; x'z \cdot 1 = x'z$

Summary of Boolean Algebra

	Property	Dual Property
Identity	$x + 0 = x$	$x \cdot 1 = x$
Complement	$x + x' = 1$	$x \cdot x' = 0$
Null	$x + 1 = 1$	$x \cdot 0 = 0$
Idempotence	$x + x = x$	$x \cdot x = x$
Involution	$(x')' = x$	
Commutative	$x + y = y + x$	$x\,y = y\,x$
Associative	$(x + y) + z = x + (y + z)$	$(x\,y)\,z = x\,(y\,z)$
Distributive	$x\,(y + z) = xy + xz$	$x + yz = (x + y)(x + z)$
Absorption	$x + xy = x$	$x(x + y) = x$
Simplification	$x + x'y = x + y$	$x(x' + y) = xy$
DeMorgan	$(x + y)' = x'\,y'$	$(x\,y)' = x' + y'$

Duality Principle

- The **dual** of a Boolean expression can be obtained by:
  - Interchanging AND ($\cdot$) and OR (+) operators
  - Interchanging 0's and 1's
- Example: the dual of $x(y + z')$ is $x + yz'$
  - The complement operator does not change
- The properties of Boolean algebra appear in **dual pairs**
  - If a property is proven to be true then its dual is also true

	Property	Dual Property
Identity	$x + 0 = x$	$x \cdot 1 = x$
Complement	$x + x' = 1$	$x \cdot x' = 0$
Distributive	$x\,(y + z) = xy + xz$	$x + yz = (x + y)(x + z)$

Expression Simplification

- Using Boolean algebra to simplify expressions
- Expression should contain the smallest number of literals
- A literal is a variable that may or may not be complemented
- **Example:** Simplify the following Boolean function to a minimum number of literals.

$$F(A, C) = (A + C)' + (A + C)(A' + C')$$

- **Solution:** $(A + C)' + (A + C)(A' + C')$ (6 literals)

$= (A + C)' + (A' + C')$ by simplification (4 literals)

$= A'C' + A' + C'$ by DeMorgan (4 literals)

$= A' + C'$ by absorption (2 literals)

Expression Simplification

- **Example:** Simplify $A'B' + B'C + AB'C' + AB$ to a minimum number of literals
- **Solution:** $A'B' + B'C + AB'C' + AB$ (9 literals)

$= B'(A' + C + AC') + AB$ by distributive (7 literals)

$= B'(A' + C + A) + AB$ by simplification (6 literals)

$= B'(C + 1) + AB$ by complement (4 literals)

$= B' + AB$ by null and identity (3 literals)

$= B' + A$ by simplification (2 literals)

Expression Simplification

- **Example:** Simplify $ab + a'cd + a'bd + a'cd' + abcd$ to a minimum number of literals
- **Solution:** $ab + a'cd + a'bd + a'cd' + abcd$ (15 literals)

$= ab + abcd + a'cd + a'cd' + a'bd$ by commutative (15 literals)

$= ab(1 + cd) + a'c(d + d') + a'bd$ by distributive (11 literals)

$= ab + a'c + a'bd$ by complement, null, and identity (7 literals)

$= ba + ba'd + a'c$ by commutative (7 literals)

$= b(a + a'd) + a'c$ by distributive (6 literals)

$= b(a + d) + a'c$ by simplification (5 literals)

Expression Simplification

- **Example:** Simplify $(A'+B'+C')(A+C')(B+C')(B'+C)$ to a minimum number of literals
- **Solution:** $(A'+B'+C')(A+C')(B+C')(B'+C)$ (9 literals)

$= A'B'C' + AC' + BC' + B'C$ by dual (9 literals)

$= C'(A'B' + A + B) + B'C$ by distributive (7 literals)

$= C'(A + B' + B) + B'C$ by simplification (6 literals)

$= C'(A + 1) + B'C$ by complement (4 literals)

$= C' + B'C$ by null and identity (3 literals)

$= C' + B'$ by simplification (2 literals)

Then, we take the dual again, this leads to

$= C'B'$

Dual and Complementing Boolean Functions

- The complement of a function can be achieved by taking its dual and complementing each literal

- **Example:** what is the complement of $f = x'yz' + xy'z'$?

  The dual $(x' + y + z')(x + y' + z')$

  Complement each literal $(x + y' + z)(x' + y + z) = f'$

- **Example:** what is the complement of $g = (a' + bc)d' + e$?

  The dual $\left(\left(a'.(b + c)\right) + d'\right).e$

  Complement each literal $\left(\left(a.(b' + c')\right) + d\right).e'$

  $(a(b' + c') + d)e' = g'$

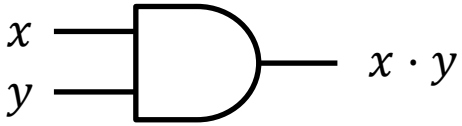
Importance of Boolean Algebra

- Our objective is to learn how to design digital circuits
- These circuits use signals with two possible values
- Logic **0** is a **low** voltage signal (around 0 volts)
- Logic **1** is a **high** voltage signal (e.g. 5 or 3.3 volts)
- The physical value of a signal is the actual voltage it carries, while its logic value is either 0 (low) or 1 (high)
- Having only two logic values (0 and 1) simplifies the implementation of the digital circuit

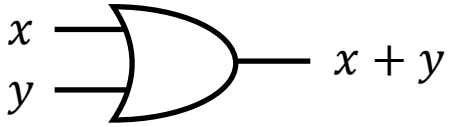
Next . . .

- Boolean Algebra
- Boolean Functions and Truth Tables
- DeMorgan's Theorem
- Algebraic manipulation and expression simplification
- **Logic gates and logic diagrams**
- **Minterms and Maxterms**
- **Sum-Of-Products and Product-Of-Sums**
- Additional Gates and Symbols

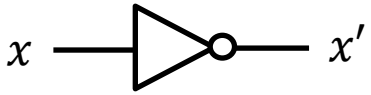
Logic Gates and Symbols



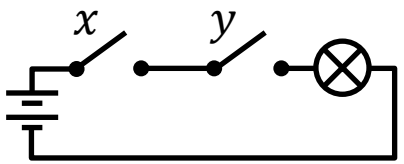
AND gate



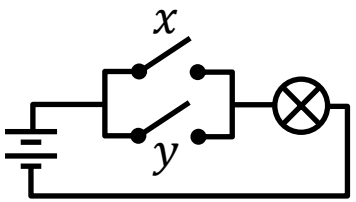
OR gate



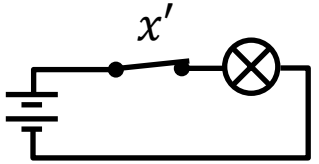
NOT gate (inverter)



AND: Switches in series
logic 0 is open switch



OR: Switches in parallel
logic 0 is open switch



NOT: Switch is normally
closed when x is 0

- In the earliest computers, relays were used as mechanical switches controlled by electricity (coils)
- Today, tiny transistors are used as electronic switches that implement the logic gates (CMOS technology)

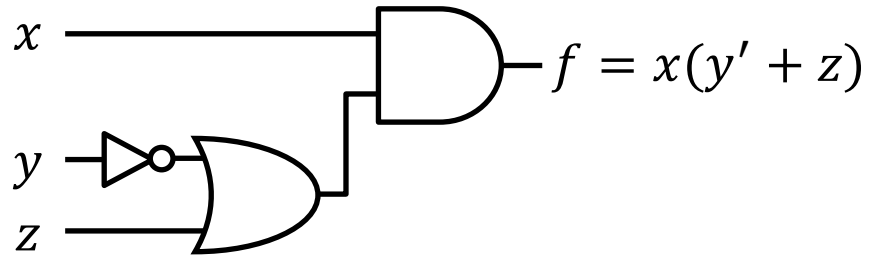
Truth Table and Logic Diagram

- Given the following logic function: $f = x(y' + z)$
- Draw the corresponding truth table and logic diagram

Truth Table

x y z	y'+z	f = x(y'+z)
0 0 0	1	0
0 0 1	1	0
0 1 0	0	0
0 1 1	1	0
1 0 0	1	1
1 0 1	1	1
1 1 0	0	0
1 1 1	1	1

Logic Diagram



Truth Table and Logic Diagram describe the same function f. Truth table is unique, but logic expression and logic diagram are not. This gives flexibility in implementing logic functions.

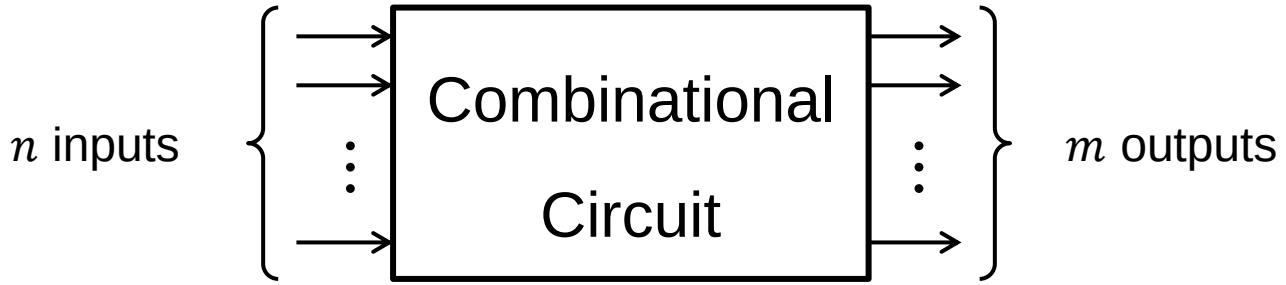
Combinational Circuit

- A combinational circuit is a block of logic gates having:

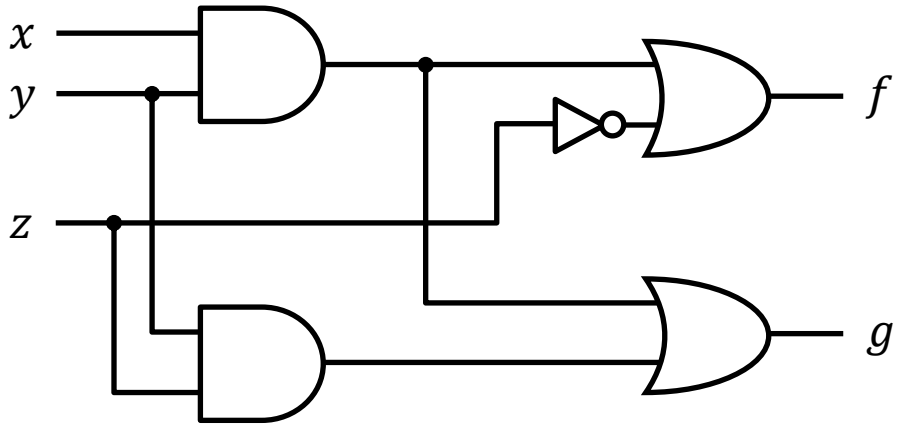
  n inputs: $x_1, x_2, \ldots, x_n$

  m outputs: $f_1, f_2, \ldots, f_m$

- Each output is a function of the input variables
- Each output is determined from present combination of inputs
- Combination circuit performs operation specified by logic gates



Example of a Simple Combinational Circuit



- The above circuit has:
  - Three inputs: x, y, and z
  - Two outputs: f and g
- What are the logic expressions of f and g ?
- **Answer:** $f = xy + z'$

  $g = xy + yz$

From Truth Tables to Gate Implementation

- Given the truth table of a Boolean function f, how do we implement the truth table using logic gates?

Truth Table

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

What is the logic expression of f?

What is the gate implementation of f?

To answer these questions, we need to define Minterms and Maxterms

Minterms and Maxterms

- **Minterms** are AND terms with every variable present in either true or complement form
- **Maxterms** are OR terms with every variable present in either true or complement form

Minterms and Maxterms for 2 variables x and y

x	y	index	Minterm	Maxterm
0	0	0	$m_0 = x'y'$	$M_0 = x + y$
0	1	1	$m_1 = x'y$	$M_1 = x + y'$
1	0	2	$m_2 = xy'$	$M_2 = x' + y$
1	1	3	$m_3 = xy$	$M_3 = x' + y'$

- For *n* variables, there are 2^n Minterms and Maxterms

Minterms and Maxterms for 3 Variables

x	y	z	index	Minterm	Maxterm
0	0	0	0	$m_0 = x'y'z'$	$M_0 = x + y + z$
0	0	1	1	$m_1 = x'y'z$	$M_1 = x + y + z'$
0	1	0	2	$m_2 = x'yz'$	$M_2 = x + y' + z$
0	1	1	3	$m_3 = x'yz$	$M_3 = x + y' + z'$
1	0	0	4	$m_4 = xy'z'$	$M_4 = x' + y + z$
1	0	1	5	$m_5 = xy'z$	$M_5 = x' + y + z'$
1	1	0	6	$m_6 = xyz'$	$M_6 = x' + y' + z$
1	1	1	7	$m_7 = xyz$	$M_7 = x' + y' + z'$

Maxterm M_i is the **complement** of Minterm m_i

$$M_i = m_i' \text{ and } m_i = M_i'$$

Purpose of the Index

- Minterms and Maxterms are designated with an index
- The **index** for the Minterm or Maxterm, expressed as a **binary number**, is used to determine whether the variable is shown in the true or complemented form
- For Minterms:
  - '**1**' means the variable is **Not Complemented**
  - '**0**' means the variable is **Complemented**
- For Maxterms:
  - '**0**' means the variable is **Not Complemented**
  - '**1**' means the variable is **Complemented**

Sum-Of-Minterms (SOM) Canonical Form

Truth Table

x y z	f	Minterm
0 0 0	0	
0 0 1	0	
0 1 0	1	$m_2 = x'yz'$
0 1 1	1	$m_3 = x'yz$
1 0 0	0	
1 0 1	1	$m_5 = xy'z$
1 1 0	0	
1 1 1	1	$m_7 = xyz$

Sum of Minterm entries that evaluate to '**1**'

Focus on the '**1**' entries

$$f = m_2 + m_3 + m_5 + m_7$$

$$f = \sum(2, 3, 5, 7)$$

$$f = x'yz' + x'yz + xy'z + xyz$$

Example - Sum-Of-Minterms

- Express the Boolean function $\boldsymbol{F(A,B,C) = A + B'C}$ as a sum of minterms.
  - The first term A is missing two variables; therefore,

$$A = A(B + B') = AB + AB'$$
$$= AB(C + C') + AB'(C + C') = ABC + ABC' + AB'C + AB'C'$$

  - The second term $B'C$ is missing one variables; therefore,

$$B'C = (A + A')B'C = AB'C + A'B'C$$

  - Combining all terms (and remove duplicate terms), we have

$$F = A + B'C = ABC + ABC' + AB'C + AB'C' + A'B'C$$
$$= m_7 + m_6 + m_5 + m_4 + m_1$$

$$F(A,B,C) = \sum(1,4,5,6,7)$$

Example - Sum-Of-Minterms

- Express the Boolean function $F(A, B, C) = A + B'C$ as a sum of minterms.
  - An alternative procedure is to obtain the truth table of the function directly from the algebraic expression and then read the minterms from the truth table.

A B C	F	Minterm
0 0 0	0	
0 0 1	1	$m_1 = A'B'C$
0 1 0	0	
0 1 1	0	
1 0 0	1	$m_4 = AB'C'$
1 0 1	1	$m_5 = AB'C$
1 1 0	1	$m_6 = ABC'$
1 1 1	1	$m_7 = ABC$

$$F(A, B, C) = \sum (1, 4, 5, 6, 7)$$

Example - Sum-Of-Minterms

- Express $f(a,b,c,d) = \sum(2,3,6,10,11)$ in the sum-of-minterms form
  - $f(a,b,c,d) = m_2 + m_3 + m_6 + m_{10} + m_{11}$
  - $f(a,b,c,d) = a'b'cd' + a'b'cd + a'bcd' + ab'cd' + ab'cd$
- Express $g(a,b,c,d) = \sum(0,1,12,15)$ in the sum-of-minterms form
  - $g(a,b,c,d) = m_0 + m_1 + m_{12} + m_{15}$
  - $g(a,b,c,d) = a'b'c'd' + a'b'c'd + abc'd' + abcd$

Product-Of-Maxterms (POM) Canonical Form

Truth Table

x y z	f	Maxterm
0 0 0	0	$M_0 = x + y + z$
0 0 1	0	$M_1 = x + y + z'$
0 1 0	1	
0 1 1	1	
1 0 0	0	$M_4 = x' + y + z$
1 0 1	1	
1 1 0	0	$M_6 = x' + y' + z$
1 1 1	1	

Product of Maxterm entries that evaluate to '**0**'

Focus on the '**0**' entries

$$f = M_0 \cdot M_1 \cdot M_4 \cdot M_6$$

$$f = \prod(0, 1, 4, 6)$$

$$f = (x + y + z)(x + y + z')(x' + y + z)(x' + y' + z)$$

Example - Product-Of-Maxterms

- Express the Boolean function $f(x, y, z) = xy + x'z$ as a product of maxterms.
  - Convert the function into OR terms by using the distributive law,

$$f = xy + x'z = (xy + x')(xy + z) = (x' + x)(x' + y)(x + z)(y + z)$$
$$= (x' + y)(x + z)(y + z)$$

  - Each term is missing one variables; therefore,

$$f = (x' + y)(x + z)(y + z) = (x' + y + zz')(x + yy' + z)(xx' + y + z)$$
$$= (x' + y + z)(x' + y + z')(x + y + z)(x + y' + z)(x + y + z)(x' + y + z)$$

  - Removing duplicate terms, we have

$$f = (x' + y + z)(x' + y + z')(x + y + z)(x + y' + z)$$
$$= M_4 \cdot M_5 \cdot M_0 \cdot M_2$$
$$f(x, y, z) = \prod(0, 2, 4, 5)$$

Example - Product-Of-Maxterms

- Express the Boolean function $f(x, y, z) = xy + x'z$ as a product of maxterms.
  - An alternative procedure is to obtain the truth table of the function directly from the algebraic expression and then read the maxterms from the truth table

$$F(x, y, z) = \prod(0, 2, 4, 5)$$

x y z	f	Maxterm
0 0 0	0	$M_0 = A + B + C$
0 0 1	1	
0 1 0	0	$M_2 = A + B' + C$
0 1 1	1	
1 0 0	0	$M_4 = A' + B + C$
1 0 1	0	$M_5 = A + B' + C$
1 1 0	1	
1 1 1	1	

Examples - Product-Of-Maxterms

- Express $f(a,b,c,d) = \prod(1,3,11)$ in the product-of-maxterms form
  - $f(a,b,c,d) = M_1 \cdot M_3 \cdot M_{11}$
  - $f(a,b,c,d) = (a+b+c+d')(a+b+c'+d')\,(a'+b+c'+d')$
- Express $g(a,b,c,d) = \prod(0,5,13)$ in the product-of-maxterms form
  - $g(a,b,c,d) = M_0 \cdot M_5 \cdot M_{13}$
  - $g(a,b,c,d) = (a+b+c+d)(a+b'+c+d')\,(a'+b'+c+d')$

Conversions between Canonical Forms

- The same Boolean function f can be expressed in two ways:
  - Sum-of-Minterms $f = m_0 + m_2 + m_3 + m_5 + m_7 = \sum(0, 2, 3, 5, 7)$
  - Product-of-Maxterms $f = M_1 \cdot M_4 \cdot M_6 = \prod(1, 4, 6)$

Truth Table

x	y	z	f	Minterms	Maxterms
0	0	0	1	$m_0 = x'y'z'$	
0	0	1	0		$M_1 = x + y + z'$
0	1	0	1	$m_2 = x'yz'$	
0	1	1	1	$m_3 = x'yz$	
1	0	0	0		$M_4 = x' + y + z$
1	0	1	1	$m_5 = xy'z$	
1	1	0	0		$M_6 = x' + y' + z$
1	1	1	1	$m_7 = xyz$	

To convert from one canonical form to another, interchange the symbols $\sum$ and $\prod$ and list those numbers missing from the original form.

Function Complement

Truth Table

x	y	z	f	f'
0	0	0	1	0
0	0	1	0	1
0	1	0	1	0
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0

Given a Boolean function f

$$f(x,y,z) = \sum(0,2,3,5,7) = \prod(1,4,6)$$

Then, the complement f' of function f

$$f'(x,y,z) = \prod(0,2,3,5,7) = \sum(1,4,6)$$

The complement of a function expressed by a Sum of Minterms is the Product of Maxterms with the same indices. Interchange the symbols $\sum$ and $\prod$, but keep the same list of indices.

Example

- Write the complement of the following function using sum of minterms

$$f(z, y, x) = \sum(0, 2, 3, 4, 6)$$

  - Since the system has 3 input variables (z, y, x), the number of minterms and maxterms = $2^3 = 8$

$$f(z, y, x) = \sum(0, 2, 3, 4, 6)$$

$$f'(z, y, x) = \prod(0, 2, 3, 4, 6) = \sum(1, 5, 7)$$

$$= z'y'x + zy'x + zyx$$

Example

- Given that $f(x, y, z, w) = \sum(0, 1, 2, 4, 5, 7)$, derive the product of maxterms expression of f and the two canonical form expressions of f'
  - Since the system has 4 input variables (x,y,z,w), the number of minterms and maxterms = $2^4 = 16$

$$f(x, y, z, w) = \sum(0, 1, 2, 4, 5, 7)$$
$$f(x, y, z, w) = \prod(3, 6, 8, 9, 10, 11, 12, 13, 14, 15)$$

$$f'(x, y, z, w) = \sum(3, 6, 8, 9, 10, 11, 12, 13, 14, 15)$$
$$f'(x, y, z, w) = \prod(0, 1, 2, 4, 5, 7)$$

Summary of Minterms and Maxterms

- There are 2^n Minterms and Maxterms for Boolean functions with n variables, indexed from 0 to $2^n - 1$
- Minterms correspond to the **1-entries** of the function
- Maxterms correspond to the **0-entries** of the function
- Any Boolean function can be expressed as a Sum-of-Minterms and as a Product-of-Maxterms
- For a Boolean function, given the list of Minterm indices one can determine the list of Maxterms indices (and vice versa)
- The complement of a Sum-of-Minterms is a Product-of-Maxterms with the same indices (and vice versa)

Operations on Functions

- The AND operation on two functions corresponds to the intersection of the two sets of minterms of the functions
- The OR operation on two functions corresponds to the union of the two sets of minterms of the functions
- **Example:** Let $F(A, B, C) = \Sigma m(1, 3, 6, 7)$ and $G(A, B, C) = \Sigma m(0, 1, 2, 4, 6, 7)$
  - $F.G = \Sigma m(1, 6, 7)$
  - $F + G = \Sigma m(0, 1, 2, 3, 4, 6, 7)$
  - $F'.G = ??$
    - $F' = \Sigma m(0, 2, 4, 5)$
    - $F'.G = \Sigma m(0, 2, 4)$

Equal Functions

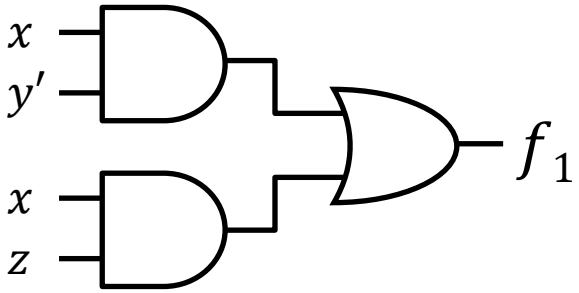
- Two functions are equal **if and only if** they have the same sum of minterms and the same product of maxterms.
- **Example:** Are the function $\mathbf{F_1 = a'b' + ac + bc'}$ and the function $\mathbf{F_2 = a'c' + ab + b'c}$ equal?
  - $F_1 = \Sigma m(0, 1, 2, 5, 6, 7) = \prod(3, 4)$
  - $F_2 = \Sigma m(0, 1, 2, 5, 6, 7) = \prod(3, 4)$
  - Thus, they are equal.
- **Example:** Are the function $\mathbf{F_1(x, y, z) = \Sigma m(1, 2, 4, 5, 6, 7)}$ and the function $\mathbf{F_2(a, b) = \prod(0, 3)}$ equal?
  - $F_1 = \Sigma m(1, 2, 4, 5, 6, 7) = \prod(0, 3)$
  - $F_2 = \prod(0, 3) = \Sigma m(1, 2)$
  - Thus, they are not equal.

Sum-of-Products and Products-of-Sums

- Canonical forms contain a larger number of literals
  - Because the Minterms (and Maxterms) must contain, by definition, all the variables either complemented or not
- Another way to express Boolean functions is in **standard** form
- Two standard forms: **Sum-of-Products** and **Product-of -Sums**
- Sum of Products (**SOP**)
  - Boolean expression is the **ORing** (sum) of **AND terms** (products)
  - Examples: $f_1 = xy' + xz$ $\qquad$ $f_2 = y + xy'z$
- Products of Sums (**POS**)
  - Boolean expression is the **ANDing** (product) of **OR terms** (sums)
  - Examples: $f_3 = (x + z)(x' + y')$ $\qquad$ $f_4 = x(x' + y' + z)$

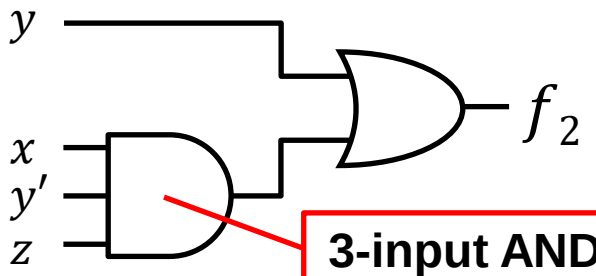
Two-Level Gate Implementation

$f_1 = xy' + xz$



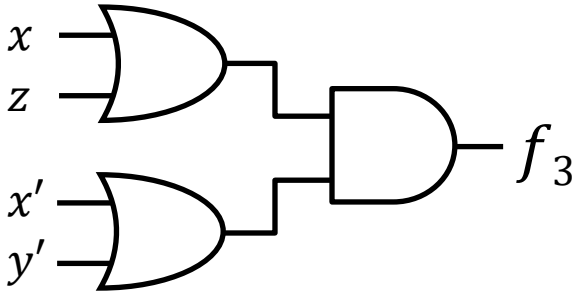
AND-OR implementations

$f_2 = y + xy'z$



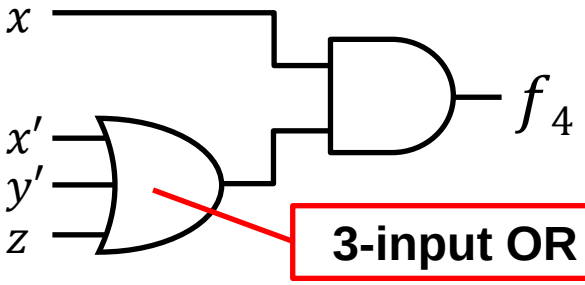
3-input AND gate

$f_3 = (x + z)(x' + y')$



OR-AND implementations

$f_4 = x(x' + y' + z)$



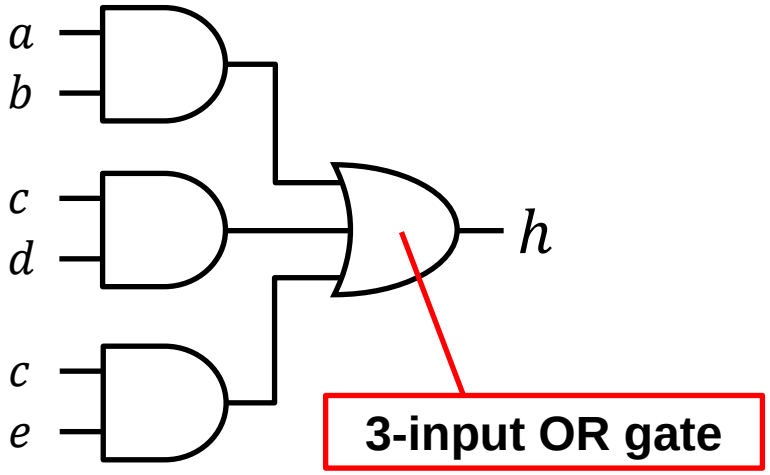
3-input OR gate

Two-Level vs. Three-Level Implementation

- $h = ab + cd + ce$ (6 literals) is a sum-of-products
- h may also be written as: $h = ab + c(d + e)$ (5 literals)
- However, $h = ab + c(d + e)$ is a non-standard form
  - $h = ab + c(d + e)$ is not a sum-of-products nor a product-of-sums

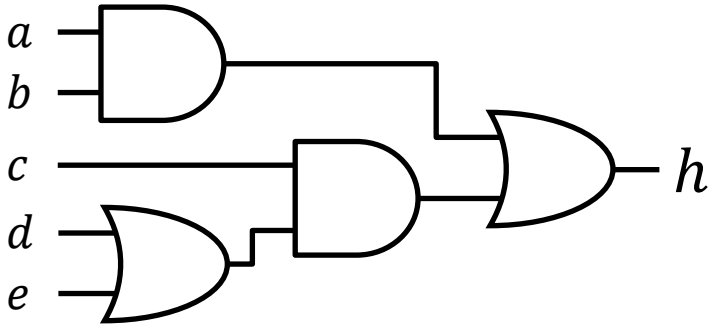
2-level implementation

$h = ab + cd + ce$



3-level implementation

$h = ab + c(d + e)$

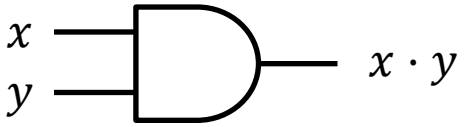


Next . . .

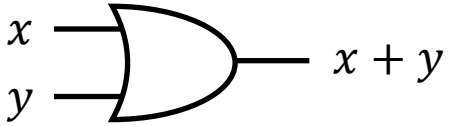
- Boolean Algebra
- Boolean Functions and Truth Tables
- DeMorgan's Theorem
- Algebraic manipulation and expression simplification
- Logic gates and logic diagrams
- Minterms and Maxterms
- Sum-Of-Products and Product-Of-Sums
- **Additional Gates and Symbols**

Additional Logic Gates and Symbols

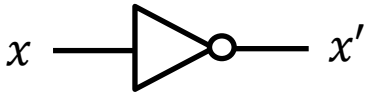
- Why?
  - Low-cost implementation
  - Useful in implementing Boolean functions



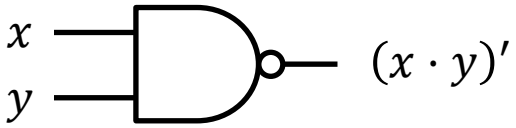
AND gate



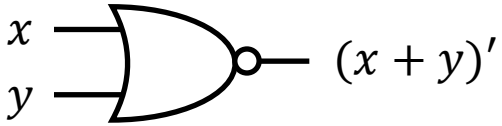
OR gate



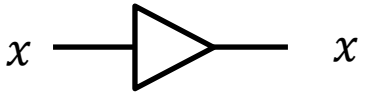
NOT gate (inverter)



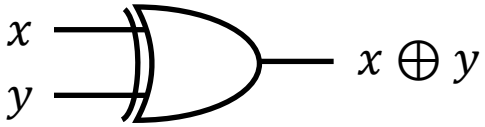
NAND gate



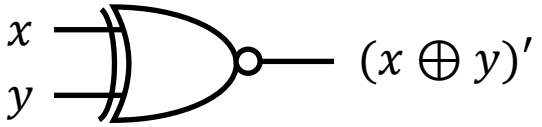
NOR gate



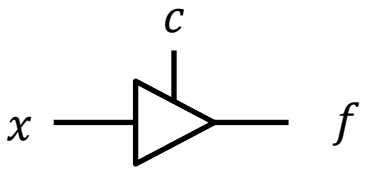
Buffer



XOR gate



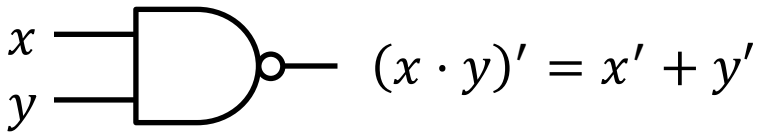
XNOR gate



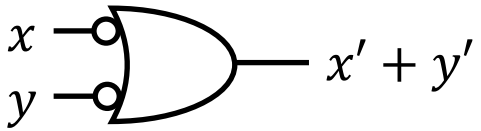
3-state gate

NAND Gate

- The NAND gate has the following symbol and truth table
- NAND represents **N**OT **AND**
- The small bubble circle represents the invert function



NAND gate



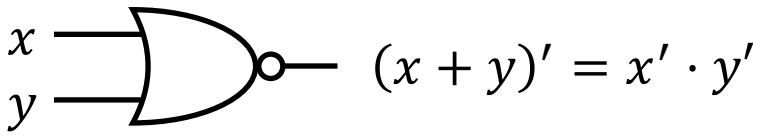
Another symbol for NAND

x	y	NAND
0	0	1
0	1	1
1	0	1
1	1	0

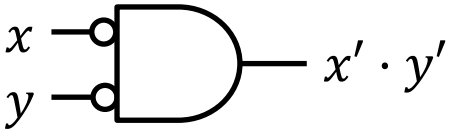
- NAND gate is implemented efficiently in CMOS technology
  - In terms of chip area and speed

NOR Gate

- The NOR gate has the following symbol and truth table
- NOR represents **N**OT **OR**
- The small bubble circle represents the invert function



NOR gate



Another symbol for NOR

x	y	NOR
0	0	1
0	1	0
1	0	0
1	1	0

- NOR gate is implemented efficiently in CMOS technology
  - In terms of chip area and speed

Non-Associative NAND / NOR Operations

- Unlike AND, NAND operation is NOT associative

  $(x \text{ NAND } y) \text{ NAND } z \neq x \text{ NAND } (y \text{ NAND } z)$

  $(x \text{ NAND } y) \text{ NAND } z = ((xy)'z)' = ((x' + y')z)' = xy + z'$

  $x \text{ NAND } (y \text{ NAND } z) = (x(yz)')' = (x(y' + z'))' = x' + yz$

- Unlike OR, NOR operation is NOT associative

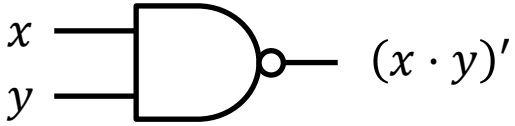
  $(x \text{ NOR } y) \text{ NOR } z \neq x \text{ NOR } (y \text{ NOR } z)$

  $(x \text{ NOR } y) \text{ NOR } z = \big((x + y)' + z\big)' = \big((x'y') + z\big)' = (x + y)z'$

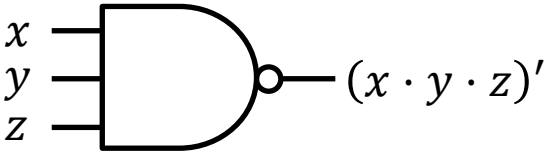
  $x \text{ NOR } (y \text{ NOR } z) = (x + (y + z)')' = \big(x + (y'z')\big)' = x'(y + z)$

Multiple-Input NAND / NOR Gates

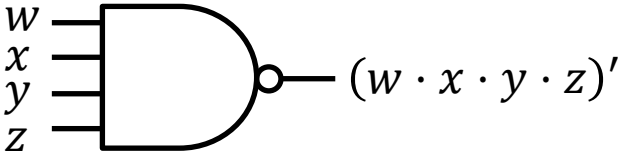
NAND/NOR gates can have multiple inputs, similar to AND/OR gates



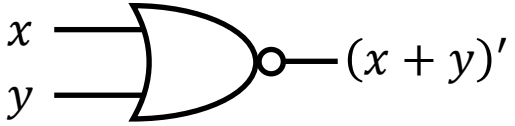
2-input NAND gate



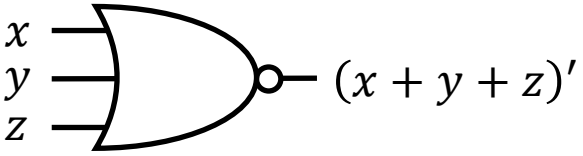
3-input NAND gate



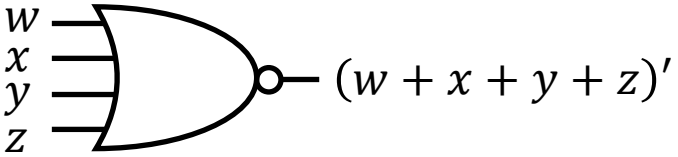
4-input NAND gate



2-input NOR gate



3-input NOR gate



4-input NOR gate

Note: a 3-input NAND is a single gate, NOT a combination of two 2-input gates.
The same can be said about other multiple-input NAND/NOR gates.

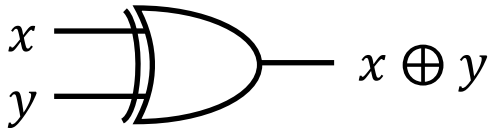
Exclusive OR / Exclusive NOR

- Exclusive OR (XOR) is an important Boolean operation used extensively in logic circuits
- Exclusive NOR (XNOR) is the complement of XOR

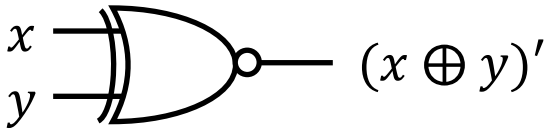
x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

x	y	XNOR
0	0	1
0	1	0
1	0	0
1	1	1

XNOR is also known as **equivalence**



XOR gate



XNOR gate

XOR / XNOR Functions

- The XOR function is: $x \oplus y = xy' + x'y$
- The XNOR function is: $(x \oplus y)' = xy + x'y'$
- XOR and XNOR gates are complex
  - Can be implemented as a true gate, or by
  - Interconnecting other gate types
- XOR and XNOR gates do not exist for more than two inputs
  - For 3 inputs, use two XOR gates
  - The cost of a 3-input XOR gate is greater than the cost of two XOR gates
- Uses for XOR and XNOR gates include:
  - Adders, subtractors, multipliers, counters, incrementers, decrementers
  - Parity generators and checkers

XOR and XNOR Properties

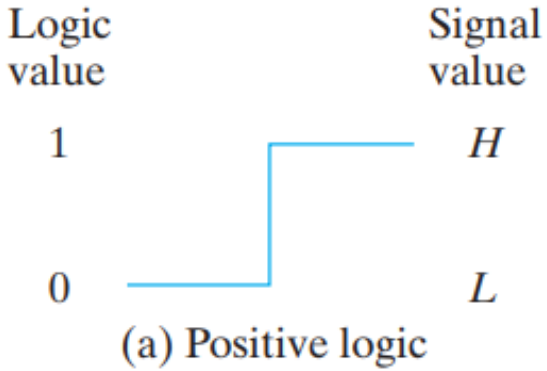
- $x \oplus 0 = x$ $\quad\quad$ $x \oplus 1 = x'$
- $x \oplus x = 0$ $\quad\quad$ $x \oplus x' = 1$
- $x \oplus y = y \oplus x$
- $x' \oplus y' = x \oplus y$
- $(x \oplus y)' = x' \oplus y = x \oplus y'$

XOR and XNOR are **associative** operations

- $(x \oplus y) \oplus z = x \oplus (y \oplus z) = x \oplus y \oplus z$
- $\big((x \oplus y)' \oplus z\big)' = (x \oplus (y \oplus z)')' = x \oplus y \oplus z$

Positive and Negative Logic

- Choosing the high-level **H** to represent logic **1** defines a positive logic system



(a) Positive logic

- Choosing the low-level **L** to represent logic **1** defines a negative logic system

(b) Negative logic

- It is up to the user to decide on a positive or negative logic polarity

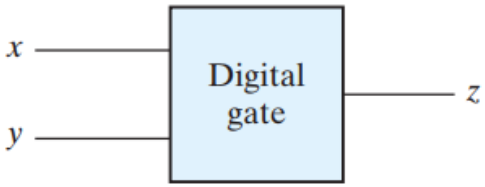
Positive and Negative Logic

- The *conversion* from positive logic to negative logic and vice versa is essentially an operation that changes 1's to 0's and 0's to 1's in both the inputs and the output of a gate

- Since this operation produces the *dual* of a function, the change of all terminals from one polarity to the other results in taking the dual of the function

x	y	z
L	L	L
L	H	L
H	L	L
H	H	H

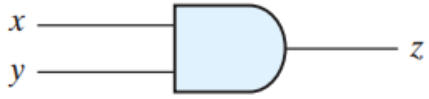
(a) Truth table with H and L



(b) Gate block diagram

x	y	z
0	0	0
0	1	0
1	0	0
1	1	1

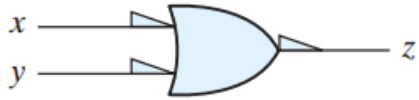
(c) Truth table for positive logic



(d) Positive logic AND gate

x	y	z
1	1	1
1	0	1
0	1	1
0	0	0

(e) Truth table for negative logic



(f) Negative logic OR gate