Chapter 2: Limits and Continuity

1. Find the following Limits

a. $\lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2}$ $\left(\frac{0}{0}\right)$

$$\lim_{t \to -1} \frac{(t+2)(t+1)}{(t+1)(t-2)} = \lim_{t \to -1} \frac{t+2}{t-2}$$

$$= \frac{1}{-3}$$

b. $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x} = \left(\frac{0}{0}\right)$

$$= \lim_{x \to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x \to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

c. $\lim_{\theta \to 1} \frac{\theta^4-1}{\theta^3-1}$ $\left(\frac{0}{0}\right)$

$$= \lim_{\theta \to 1} \frac{(\theta^2-1)(\theta^2+1)}{(\theta-1)(\theta^2+\theta+1)} = \lim_{\theta \to 1} \frac{(\theta-1)(\theta+1)(\theta^2+1)}{(\theta-1)(\theta^2+\theta+1)}$$

$$= \lim_{\theta \to 1} \frac{(\theta+1)(\theta^2+1)}{\theta^2+\theta+1} = \frac{2 \cdot 2}{3}$$

$$= \frac{4}{3}$$

d) $\lim_{\theta \to 0} \frac{\sin 2\theta}{3\theta}$ $\left(\frac{0}{0}\right)$

$= \lim_{\theta \to 0} \frac{\sin 2\theta}{3\theta} = \frac{2}{3}$.

e) $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin 2\theta}$ $\left(\frac{0}{0}\right)$

$\lim_{\theta \to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{2\sin\theta\cos\theta\,(1+\cos\theta)}$

$= \lim_{\theta \to 0} \frac{\sin\theta \cdot \sin\theta}{2\sin\theta\cos\theta\,(1+\cos\theta)}$

$= \lim_{\theta \to 0} \frac{\sin\theta}{2\cos\theta(1+\cos\theta)}$ $\quad \frac{0}{2(1)(2)} = \frac{0}{4} = 0$

f) $\lim_{x \to \infty} \frac{1+\sqrt{x}}{1-\sqrt{x}}$

$\lim_{x \to \infty} \frac{\sqrt{x}\left[\frac{1}{\sqrt{x}}+1\right]}{\sqrt{x}\left[\frac{1}{\sqrt{x}}-1\right]} = \lim_{x \to \infty} \frac{\frac{1}{\sqrt{x}}+1}{\frac{1}{\sqrt{x}}-1} = \frac{0+1}{0-1} = -1$

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(g) $\lim_{x \to -\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

Note:

$\sqrt{x^2} = |x|$

$(\sqrt{x})^2 = x$

$\lim_{x \to -\infty} \dfrac{\sqrt{x^2}\left(\sqrt{1+\frac{1}{x^2}}\right)}{x\left[1+\frac{1}{x}\right]} = \lim_{x \to -\infty} \dfrac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left[1+\frac{1}{x}\right]}$

$= \lim_{x \to -\infty} \dfrac{-x\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{1}{x}\right)}$

$= \lim_{x \to -\infty} \dfrac{-\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \dfrac{-1}{1} = -1$

(h) $\lim_{x \to -\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

$\lim_{x \to -\infty} \dfrac{x^{\frac{1}{3}} - x^{\frac{1}{5}}}{x^{\frac{1}{3}} + x^{\frac{1}{5}}}$

$= \lim_{x \to -\infty} \dfrac{x^{\frac{1}{3}}\left[1 - x^{\frac{1}{5}-\frac{1}{3}}\right]}{x^{\frac{1}{3}}\left[1 + x^{\frac{1}{5}-\frac{1}{3}}\right]} = \lim_{x \to -\infty} \dfrac{1 - x^{-\frac{2}{5}}}{1 + x^{-\frac{2}{5}}}$

$= \lim_{x \to -\infty} \dfrac{1 - \frac{1}{x^{\frac{2}{5}}}}{1 + \frac{1}{x^{\frac{2}{5}}}} = 1$

$$\lim_{t \to 3^+} \frac{\lfloor t \rfloor}{t}$$

$\lfloor t \rfloor$ Greatest integer function

$\frac{3}{3} = 1$

$\lfloor 1.3 \rfloor = 1$

$\lceil t \rceil$ = smallest Integer functin

$\lceil 1.3 \rceil = 2$

$$\lim_{x \to 0} x \sin \frac{1}{x} \qquad 0 \cdot \sin(\infty)\ ?$$

$-1 \le \sin \frac{1}{x} \le 1$

$x \to 0^+$

$-x \le x \sin \frac{1}{x} \le x \cdot 1$

$x \le x \sin \frac{1}{x} \le x$

take limit to all sides

$\lim_{x \to 0^+} x \le \lim_{x \to 0^+} x \sin \frac{1}{x} \le \lim_{x \to 0^+} x$

$0 \le \lim_{x \to 0^+} x \sin \frac{1}{x} \le 0$

$\Downarrow$

0

Sandwich Theorem

1. $g(x) \le f(x) \le h(x)$
2. $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = \alpha$

then $\lim_{x \to c} f(x) = \alpha$

same thing when $x \to 0^-$

$x \to 0^-$ $\quad -1 \le \sin \frac{1}{x} \le 1$

$-x \ge x \sin \frac{1}{x} \ge x$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$0 \qquad 0 \qquad 0$

by the two cases $\lim_{x \to 0} x \sin \frac{1}{x}$

$y=0$

② Find the asymptotes of the following functions, then sketch their graphs

ⓐ $f(x) = \frac{x+1}{x-1}$

Horizontal asymptots

"same as $x \to -\infty$"

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{x-1} = 1$

$\boxed{y=1}$

No oblique asymptots

Vertical asymptots

$x-1=0 \to x=1$

$\lim_{x\to 1^+} \frac{x+1}{x-1} = +\infty$, $\lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty$

$\boxed{x=1}$ vertical asymptots

x-int $y=0$

$0 = \frac{x+1}{x-1}$

$x+1=0 \to x=-1$

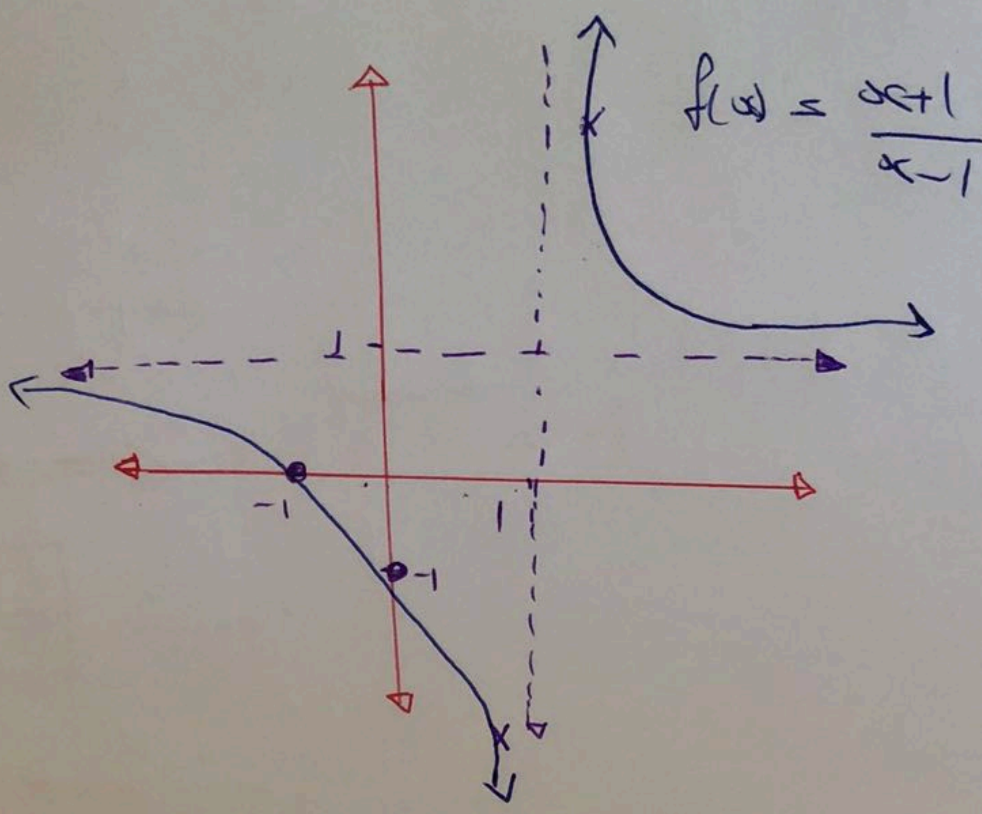
$(-1,0)$

~~y~~-int

$x=0$

$y = \frac{1}{-1} = -1$

$(0,-1)$



b) $y = \frac{x^3+1}{x^2}$

$\boxed{\text{No Horizontal Asymptote}}$

$\boxed{\text{OBlique asymptote}}$

$\boxed{y = x}$

$$\begin{array}{r|l} & x \\ x^2 & x^3+1 \\ & \underline{x^3} \\ & 1 \end{array}$$

$$\boxed{x + \frac{1}{x^2}}$$

$\boxed{\text{Vertical asymptote}}$

$x^2 = 0$

$x = 0$

$\lim_{x \to 0^+} \frac{x^3+1}{x^2} = +\infty \qquad \lim_{x \to 0^-} \frac{x^3+1}{x^2} = +\infty$

$\boxed{x = 0}$ vertical asymptote

x-int $y=0$

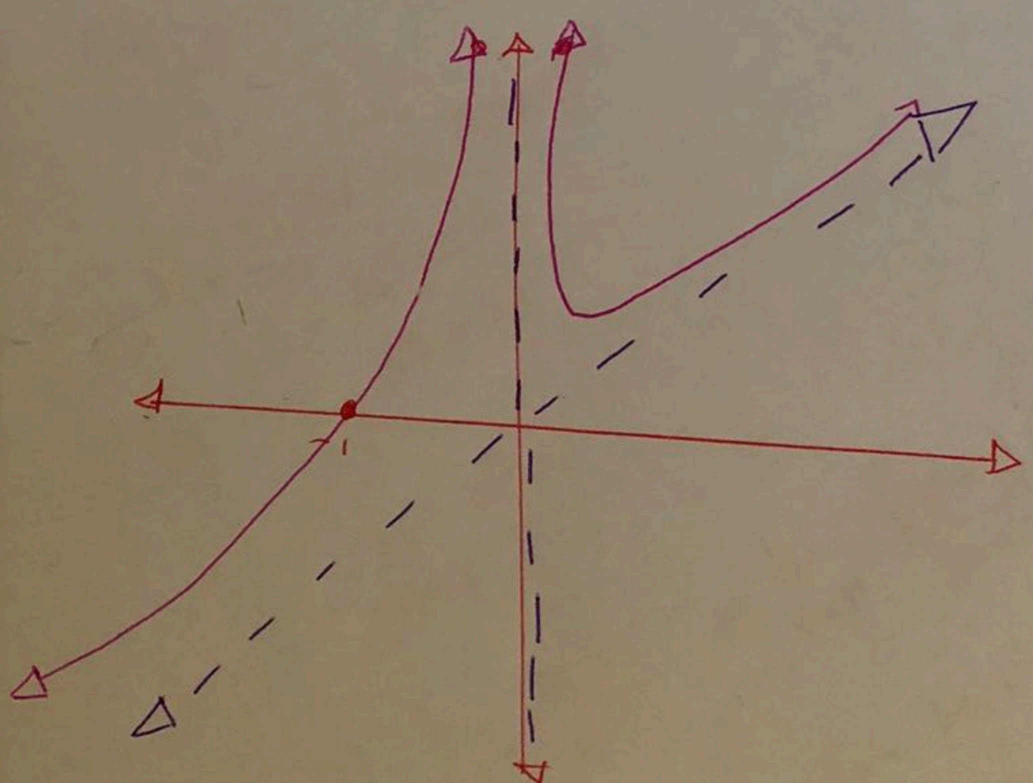
$$0=\frac{x^3+1}{x^2}$$

$$x^3=-1 \Longrightarrow x=\sqrt[3]{-1}=-1$$

$\boxed{(-1,0)}$

y-int , $x=0$

$$y=\frac{x^3+1}{x^2}$$ NO y-int



c) $f(x) = \frac{x^2+1}{x-1}$

$\boxed{\text{No Horizontal asymptote.}}$ ✓

$\boxed{\text{Oblique asymptote}}$

$\boxed{y = x+1}$ ✓

$$\begin{array}{r} x+1 \\ x-1 \overline{)\, x^2+1} \\ -\ \underline{x^2 - x} \\ x+1 \\ -\ \underline{x-1} \\ 2 \end{array}$$

$\boxed{\text{Vertical Asympta.}}$

$x=1$

$$\lim_{x\to 1^+} \frac{x^2+1}{x-1} = +\infty$$

$$\lim_{x\to 1^-} \frac{x^2+1}{x-1} = -\infty$$

So $\boxed{X=1 \text{ Vertical Asympto}}$

X-int $y=0$

$$0 = \frac{x^2+1}{x-1}$$

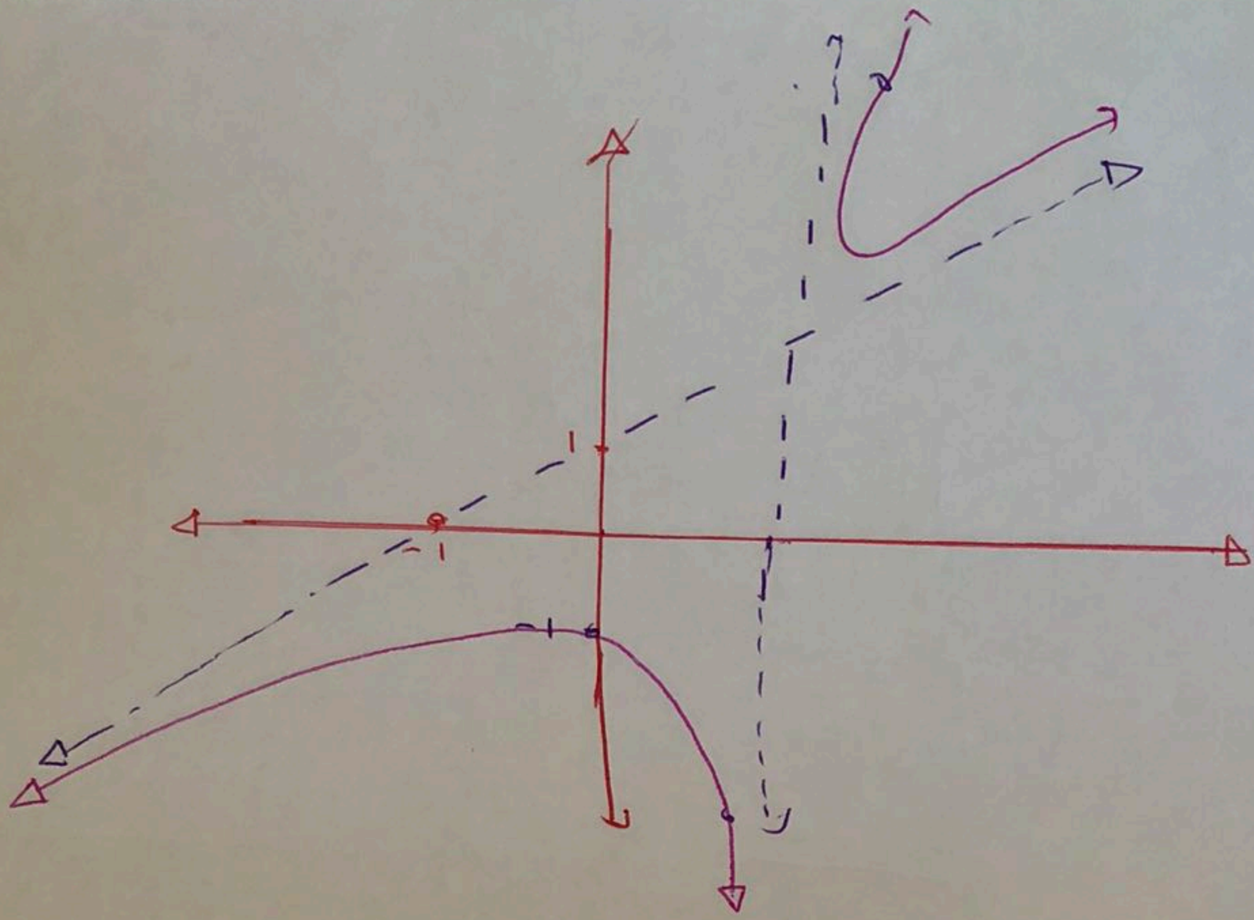
$\rightarrow x^2+1=0$

No x-int

y-int $x=0$

$$y = \frac{0+1}{0-1} = -1$$

$(0,-1)$


1
-1
-1

(d) $f(x) = \dfrac{x^3+1}{x^2-1}$

① No Horizontal asymptote

② $y = x$ Oblique Asymptote

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3+1 \\ \ominus & x^3 - x \\ \hline & x+1 \end{array}$$

③ Vertical Asymptotes

$x^2 - 1 = 0$

$x = \pm 1$

$$\lim_{x \to 1} \frac{x^3+1}{x^2-1}$$

$$\lim_{x \to 1^+} \frac{x^3+1}{x^2-1} = \lim_{x \to 1^+} \frac{\cancel{(x+1)}(x^2 - x + 1)}{\cancel{(x+1)}(x-1)} = +\infty$$

$$\lim_{x \to 1^-} \frac{x^3+1}{x^2-1} = \lim_{x \to 1^-} \frac{\cancel{(x+1)}(x^2 - x + 1)}{\cancel{(x+1)}(x-1)} = -\infty$$

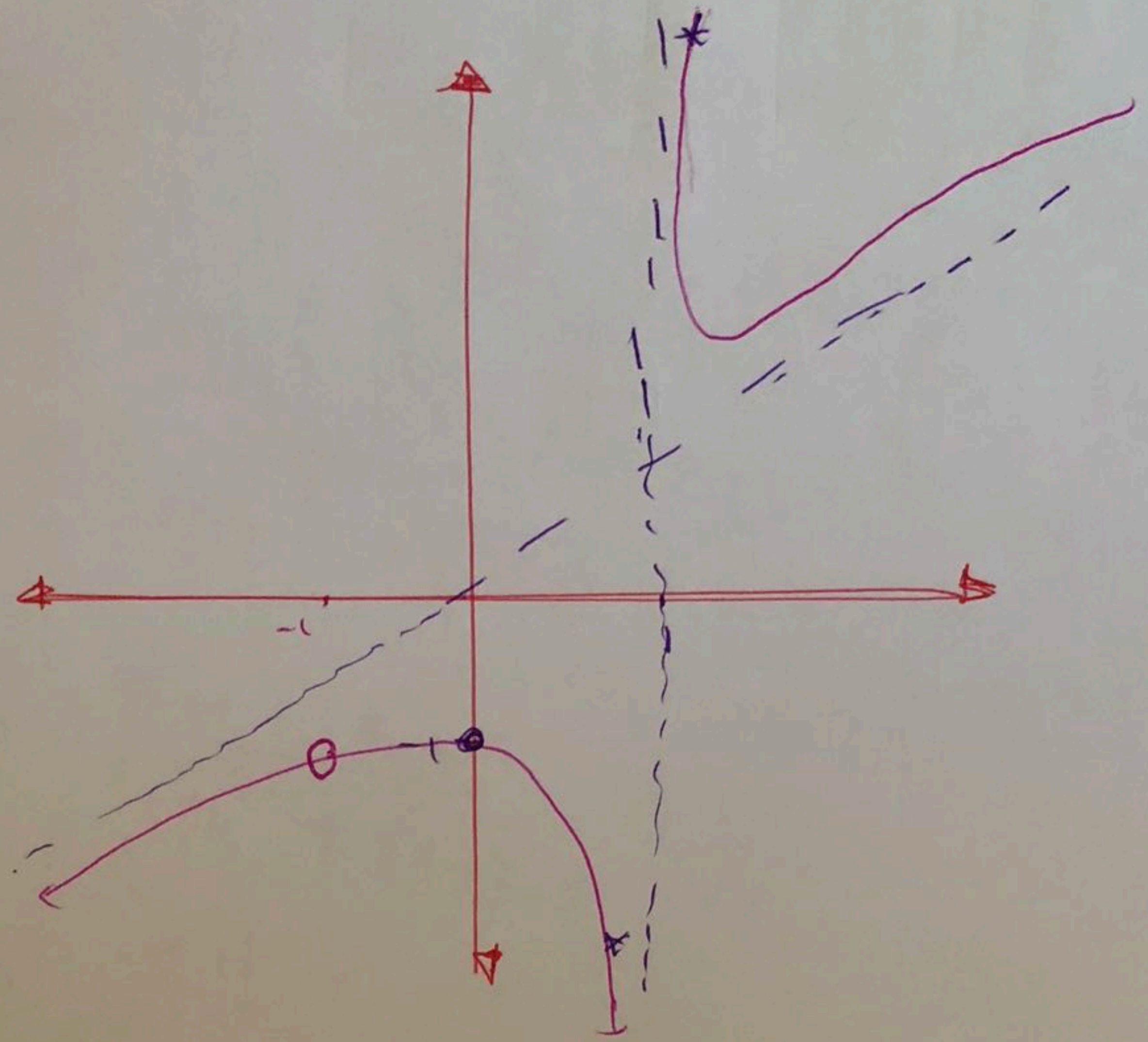
$\boxed{x = 1}$ V.A

$$\lim_{x \to -1} \frac{x^3+1}{x^2-1} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to -1^+} \frac{x^3+1}{x^2-1} = \lim_{x \to -1^+} \frac{(x+1)(x^2 - x + 1)}{(x+1)(x-1)} = \lim_{x \to -1^+} \frac{x^2 - x + 1}{x - 1} = \frac{+3}{-2}$$

$$\lim_{x \to -1^-} \frac{x^3+1}{x^2-1} = \frac{3}{-2}$$

$\boxed{x = -1 \text{ is not a V.A}}$



y-int

x-int

$x=0$

$y=0$

$y=-1$

$$0=\frac{x^3+1}{x^2-1}$$

$(0,-1)$

$x^3+1=0$

$x=-1$

No x-int

① $\lim\limits_{x\to\infty} \sqrt{x^2+1} - \sqrt{x^2-x}$

$$\lim_{x\to\infty} \frac{\left[\sqrt{x^2+1} - \sqrt{x^2-x}\right]\left[\sqrt{x^2+1} + \sqrt{x^2-x}\right]}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+1}\right)^2 - \left(\sqrt{x^2-x}\right)^2}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{x^2+1 - x^2+x}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{x\left[1+\frac{1}{x}\right]}{\sqrt{x^2}\left[\sqrt{1+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}}\right]}$$

$x\to\infty$

$\sqrt{x^2} \Rightarrow |x|$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}}} = \frac{1}{1+1} = \frac{1}{2}$$