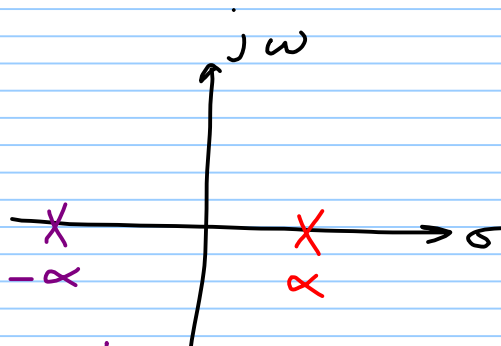


$$C(s) = R(s) G(s) = \frac{s+2}{s(s+5)} = \frac{\cancel{A} \frac{2}{5}}{s} + \frac{\cancel{B} \frac{+3}{5}}{s+5}$$

$$C(t) = \underbrace{\frac{2}{5}}_{\text{forced response}} + \underbrace{\frac{3}{5} e^{-5t}}_{\text{Natural Response}}$$

forced Response
INPUT

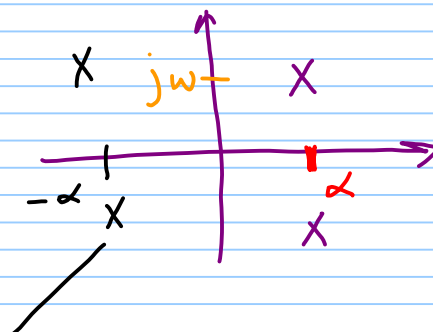
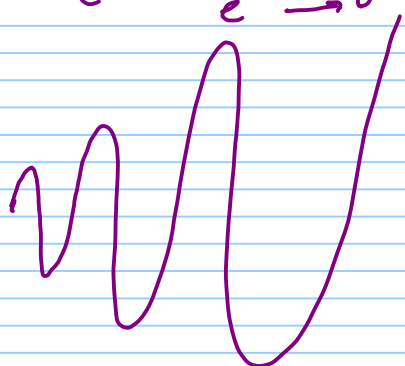
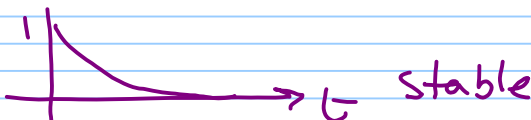
Natural Response
system (R, L, C)

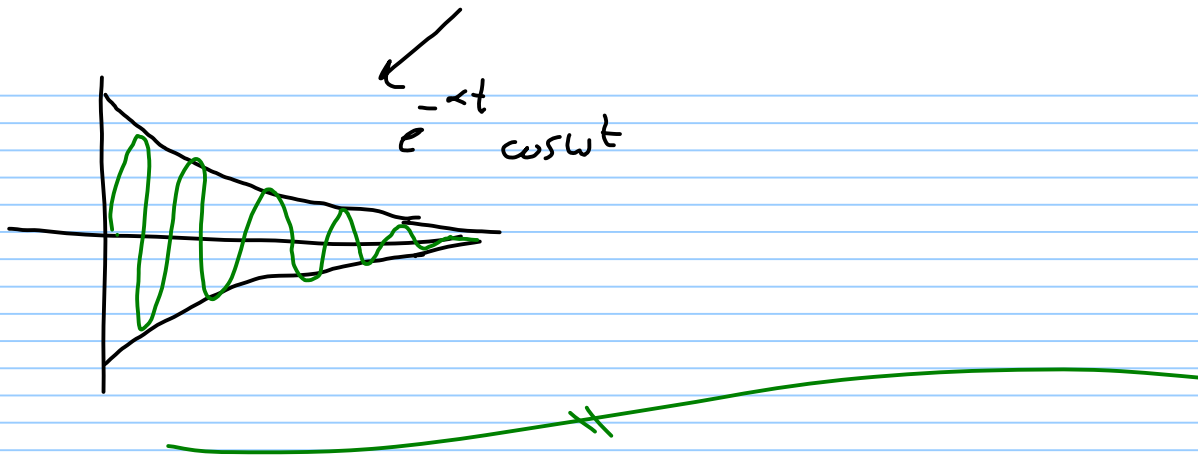


$$\left. \begin{array}{l} e^{\alpha t} \\ t \rightarrow \infty \\ e^{10t} \rightarrow \infty \end{array} \right\} \text{unstable}$$



$$\left. \begin{array}{l} e^{-\alpha t} \\ t \rightarrow \infty \\ e^{-10t} \rightarrow 0 \end{array} \right\} \text{stable}$$





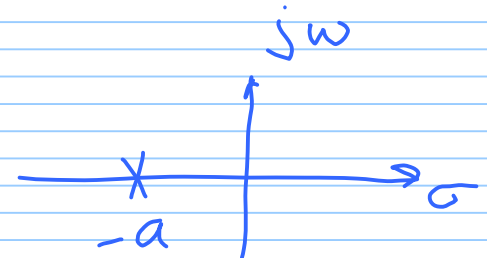
4.3 First-order systems

$\rightarrow G(s) = \frac{a}{s+a}$
 $\frac{10}{s+10}$
 $\frac{5}{s+5}$
 $\frac{1}{s+5}$

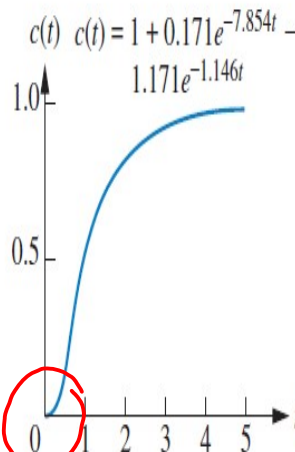
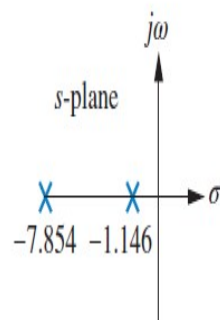
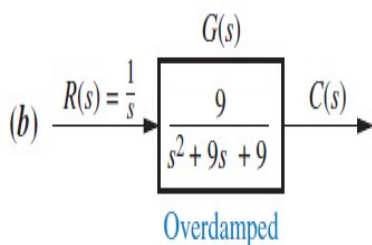
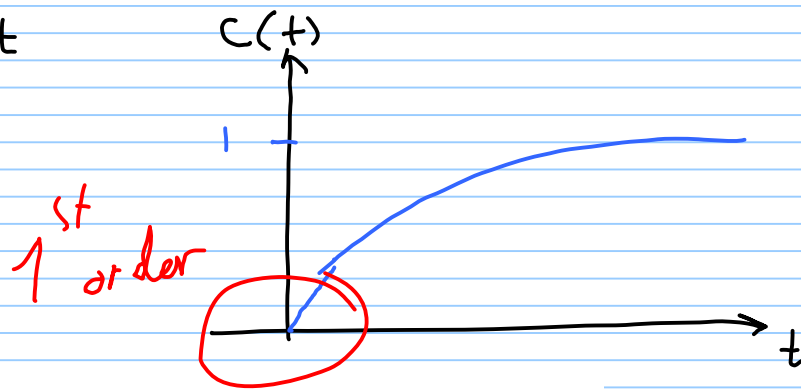
DC gain = $\frac{a}{a} = 1$

$\rightarrow$ input $\frac{1}{s}$

$\rightarrow$ output $C(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$



$c(t) = 1 - e^{-at}$



2nd order

T_s

$$C(t) = 0.98$$

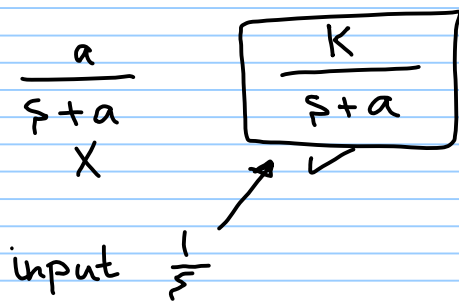
$$1 - e^{-at} = 0.98$$

$$\ln(e^{-at}) = 0.02$$

$$-at = \ln(0.02)$$

$$-at = -3.912$$

$$t = \frac{3.912}{a} \approx \frac{4}{a} = T_s$$



final value (forced response)

$$\frac{K}{a} = 0.72$$

time constant, $\frac{1}{a}$

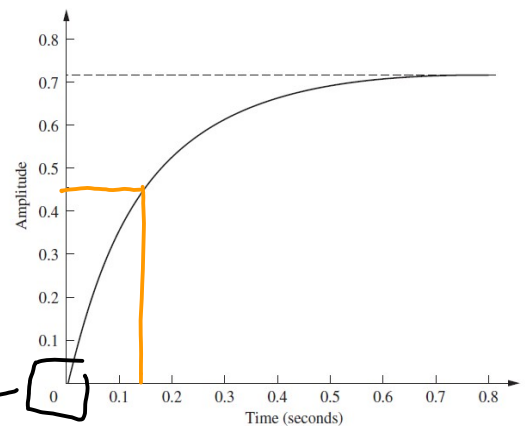
$$t_{@63\% \text{ of } 0.72} = \frac{1}{a}$$

$$K = 5.54$$

$$t_{@0.45} = \frac{1}{a}$$

$$0.13 = \frac{1}{a}$$

$$a = \frac{1}{0.13} = 7.7$$



1st Order

FIGURE: Laboratory results of a system step response test

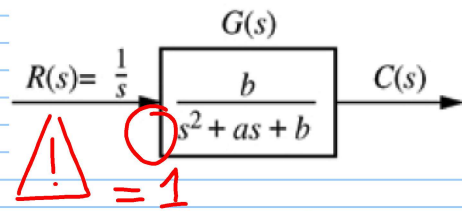
$$G(s) = \frac{5.54}{s + 7.7}$$

0.14

K=5.14

a=7.14

4.4 Second-Order Systems: Introduction



$$(b) \quad C(s) = \frac{9}{s(s+7.854)(s+1.146)}$$

$$(c) \quad C(s) = \frac{9}{s(s^2 + 2s + 9)}$$

$$\frac{9}{s(s^2 + 2s + 9)} = \frac{1}{s} + \frac{Bs + C}{s^2 + 2s + 9}$$

$$9 = \underline{s^2 + 2s + 9} + \underline{Bs^2 + Cs}$$

$$\boxed{B = -1} \quad \boxed{C = -2}$$

$$C(s) = \frac{1}{s} - \frac{s+2}{(s^2 + 2s + 1) + 9 - 1} = \frac{1}{s} - \frac{s+2}{(s+1)^2 + 8}$$

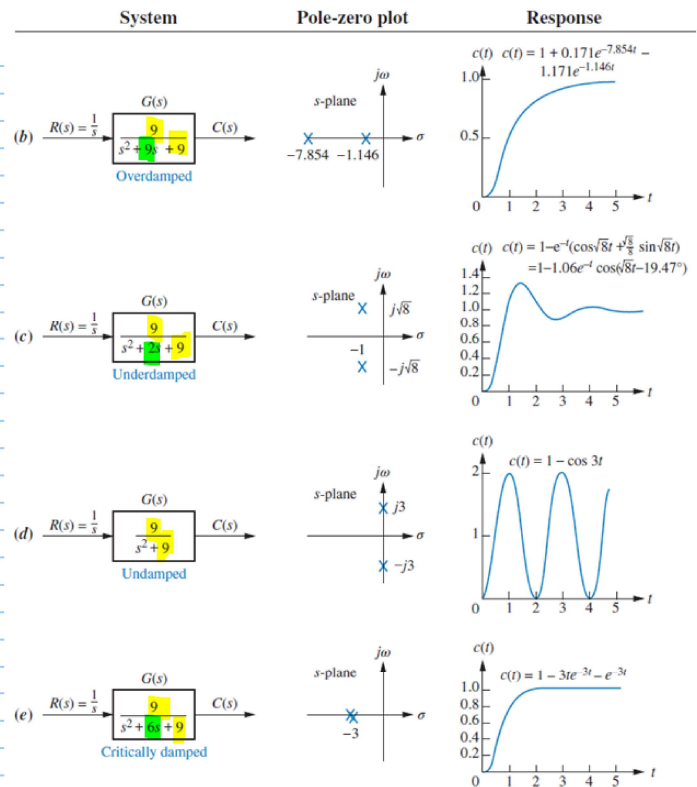
$$= \frac{1}{s} - \left[\frac{(s+1) + 1}{(s+1)^2 + (\sqrt{8})^2} \right]$$

$$= \frac{1}{s} - \frac{(s+1)}{(s+1)^2 + (\sqrt{8})^2} - \frac{1}{\sqrt{8}} \frac{\sqrt{8}}{(s+1)^2 + (\sqrt{8})^2}$$

$$c(t) = 1 - e^{-t} \cos \sqrt{8}t - \frac{1}{\sqrt{8}} e^{-t} \sin \sqrt{8}t$$

$$= 1 - e^{-t} \left[\cos \sqrt{8}t - \frac{1}{\sqrt{8}} \sin \sqrt{8}t \right]$$

$$= 1 - e^{-t} \left[\cos \sqrt{8}t - \frac{1}{\sqrt{8}} \cos(\sqrt{8}t - 90^\circ) \right]$$



$$\cos \frac{s}{s^2 + \omega^2} \rightarrow \sin \frac{\omega}{s^2 + \omega^2}$$

$$c(t) = 1 - 1.06 e^{-t} \cos(\sqrt{8}t - 19.47^\circ)$$

$$1 \angle 0^\circ - \frac{1}{\sqrt{8}} \angle -90^\circ$$

$$1 - j \frac{1}{\sqrt{8}}$$

$$1.06 \angle -19.47^\circ$$

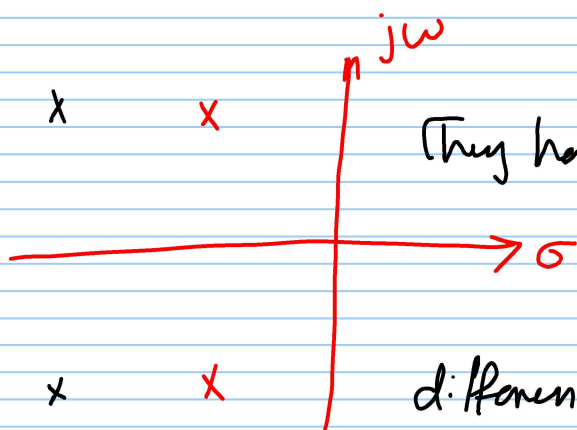
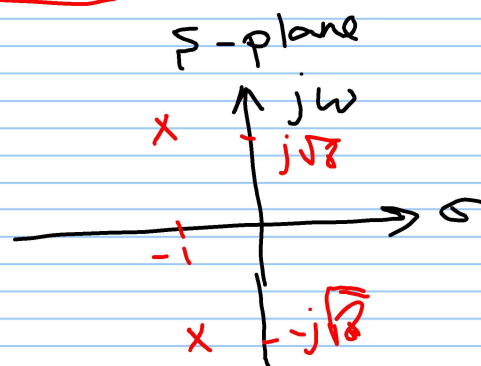
$$\begin{array}{c|c} x & +j\sqrt{8} \\ \hline x & -j\sqrt{8} \end{array}$$

$$\frac{9}{s^2 + 2s + 9}$$

Poles: -

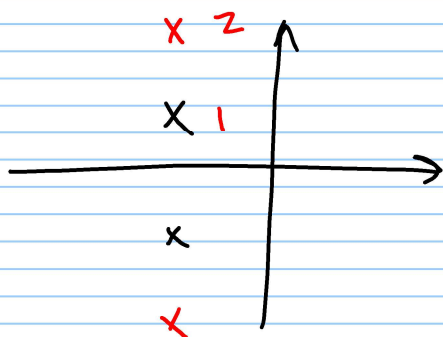
$$-2 \pm \sqrt{4 - 4 \times 9}$$

$$= -1 \pm j\sqrt{8}$$



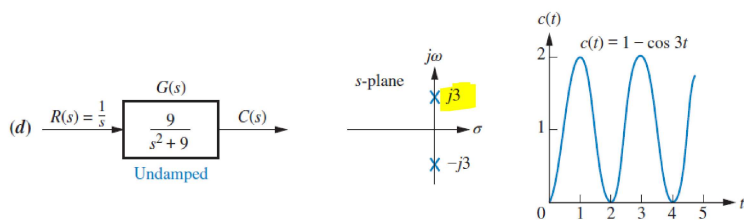
They have the same
sinusoidal freq.
 ω_d

different exponential freq. σ



$$\sigma_1 = \sigma_2$$

$$\omega_{d1} < \omega_{d2}$$



$$C(s) = \frac{9}{s(s^2 + 9)}$$

$$s_{1,2} = \pm j3$$

→ No real part

$$e^0 = 1$$

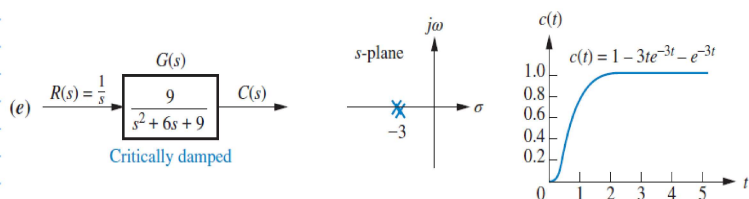
$$C(s) = \frac{1}{s} + \frac{Bs + C}{s^2 + 9}$$

$$= \frac{1}{s} - \frac{s}{s^2 + 9}$$

$$9 = s^2 + 9 + Bs^2 + Cs$$

$$B = -1, C = 0$$

$$c(t) = 1 - \cos 3t$$



$$s_{1,2} = -3$$

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s+3)^2}$$

$$\left(\frac{9}{s(s+3)^2} = \frac{1}{s} + \frac{B}{(s+3)^2} + \frac{C}{s+3} \right) s(s+3)^2$$

$$9 = s^2 + 6s + 9 + Bs^2 + C(s^2 + 3s)$$

$$1 + C = 0 \quad \boxed{C = -1}$$

$$6 + B + 3C = 0$$

$$6 + B - 3 = 0 \quad \boxed{B = -3}$$

$$C(s) = \frac{1}{s} - \frac{3}{(s+3)^2} - \frac{1}{s+3}$$

$$c(t) = 1 - 3e^{-3t} - e^{-3t}$$

4.5 The general second-order system

- Natural frequency, ω_n**

- The frequency of oscillation of the system without damping

- Damping ratio, ζ**

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/s)}} = \frac{1}{2\pi} \frac{\text{Natural period (s)}}{\text{Exponential time constant}}$$

- General TF

$$G(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where

$$a = 2\zeta\omega_n, \quad b = \omega_n^2, \quad \zeta = \frac{a}{2\omega_n}, \quad \omega_n = \sqrt{b}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$c(t) \rightarrow$ step response of the general 2nd order system

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \underbrace{s^2 + 2\zeta\omega_n s + \omega_n^2} + \underbrace{Bs^2 + Cs}$$

$$B = -1$$

$$C = -2\zeta\omega_n$$

$$\begin{aligned} C(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2} \\ &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} \end{aligned}$$

$$C(s) = \frac{1}{s} - \frac{(s + \xi\omega_n) + \xi\omega_n}{(s + \xi\omega_n)^2 + (\omega_n\sqrt{1-\xi^2})^2}$$

$$= \frac{1}{s} - \left[\frac{(s + \xi\omega_n)}{(s + \xi\omega_n)^2 + (\omega_n\sqrt{1-\xi^2})^2} + \frac{\xi}{\sqrt{1-\xi^2}} \frac{\omega_n\sqrt{1-\xi^2}}{(s + \xi\omega_n)^2 + (\omega_n\sqrt{1-\xi^2})^2} \right]$$

$$c(t) = 1 - e^{-\xi\omega_n t} \left[\cos(\omega_n\sqrt{1-\xi^2}t) + \frac{\xi}{\sqrt{1-\xi^2}} \sin(\omega_n\sqrt{1-\xi^2}t) \right]$$

$$c(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos(\omega_n\sqrt{1-\xi^2}t - \tan^{-1} \frac{\xi}{\sqrt{1-\xi^2}})$$

$$1 - j \frac{\xi}{\sqrt{1-\xi^2}}$$

$$\sqrt{1 + \frac{\xi^2}{1-\xi^2}}$$

$$\sqrt{\frac{1 - \cancel{\xi^2} + \cancel{\xi^2}}{1 - \xi^2}}$$

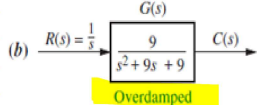
$$\frac{1}{\sqrt{1-\xi^2}}$$

$$q = 2 \times \xi \times 3$$

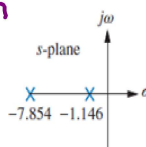
$$\omega_n = 3$$

$$\xi = \frac{9}{6} = 1.5$$

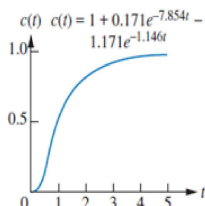
$$s^2 + 2\xi\omega_n s + \omega_n^2$$



Pole-zero plot



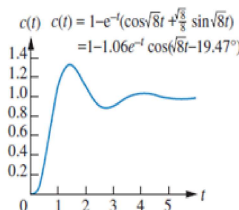
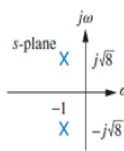
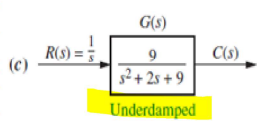
Response



$$2 \times \xi \times 3 = 2$$

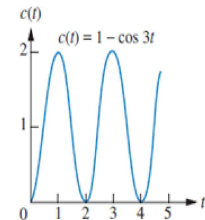
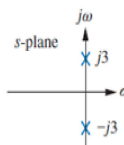
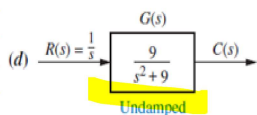
$$\omega_n = 3$$

$$\xi = \frac{1}{3} = 0.333$$



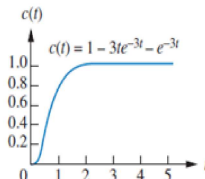
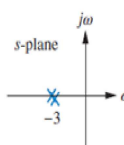
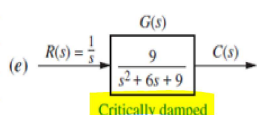
$$\omega_n = 3$$

$$\xi = 0$$



$$\omega_n = 3$$

$$\xi = 1$$

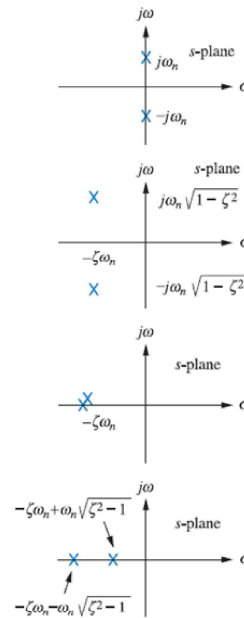


Response as a function of ζ

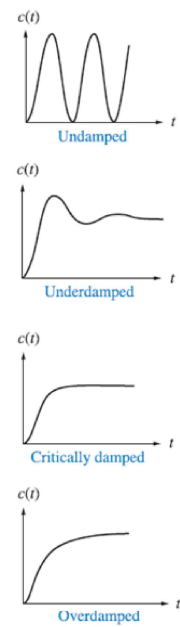
Poles

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Poles



Step response



$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\begin{aligned} s_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \end{aligned}$$

if underdamped $0 < \zeta < 1$

$$s_{1,2} = \underbrace{-\zeta\omega_n}_{\sigma_d} \pm j \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

$e^{-(\sigma_d)t}$ $\cos(\omega_d t)$

4.6 Underdamped second-order systems

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2} t - \phi\right)$$

$$\phi = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}}$$

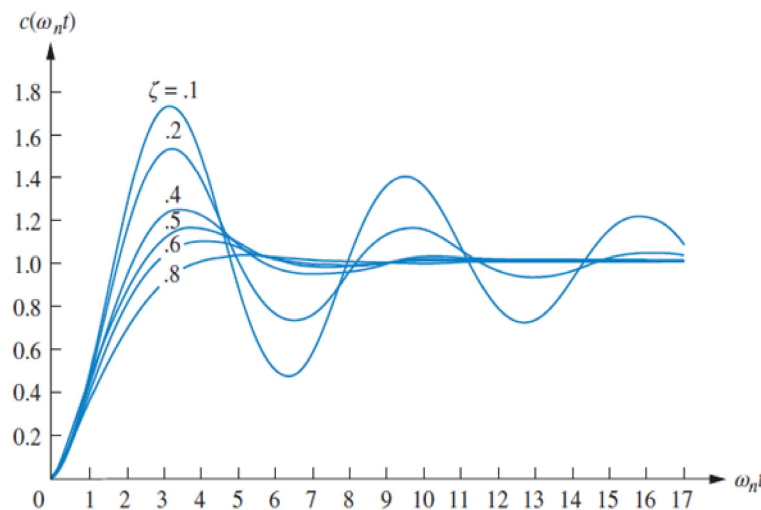
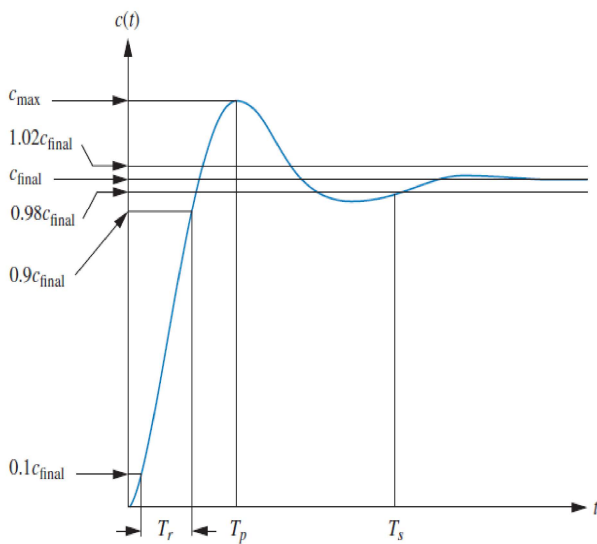


Figure: 2nd-order underdamped responses for damping ratio values



- **Rise time, T_r :** Time required for the waveform to go from 0.1 of the final value to 0.9 of the final value
- **Peak time, T_p :** Time required to reach the first, or maximum, peak
- **Overshoot, %OS:** The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady state value
- **Settling time, T_s :** Time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady state value

Evaluation of T_p peak time

$$\begin{aligned}
 \mathcal{L}\left[\frac{d}{dt}c(t)=0\right] &= s C(s) \\
 &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
 &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2} \\
 &= \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2} \\
 &= \frac{\omega_n}{\sqrt{1-\zeta^2}} \frac{\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}
 \end{aligned}$$

$$\frac{d}{dt}c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2} t)$$

$$\frac{d}{dt}c(t) = 0$$

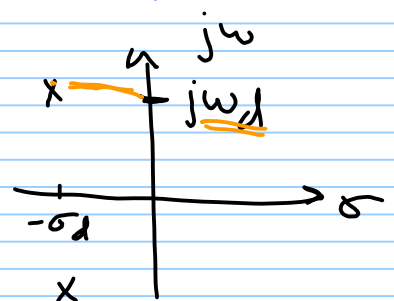
$$\omega_n\sqrt{1-\zeta^2} t = n\pi$$

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$n=0 \rightarrow t=0$
initial slope = 0

$$\boxed{n=1}$$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} = \frac{\pi}{\omega_d}$$



%OS

$$\%OS = \frac{C_{max} - C_{final}}{C_{final}} \times 100$$

$$\Rightarrow C_{final} = 1$$

$$\Rightarrow C_{max} = C(T_P)$$

$$C(t) = 1 - e^{-\xi \omega_n t} \left[\cos(\omega_n \sqrt{1-\xi^2} t) + \frac{\xi}{\sqrt{1-\xi^2}} \sin(\omega_n \sqrt{1-\xi^2} t) \right]$$

$$C_{max} = 1 - e^{-\xi \omega_n \frac{\pi}{\omega_n \sqrt{1-\xi^2}}} \left[\cos(\pi) + \frac{\xi}{\sqrt{1-\xi^2}} \sin(\pi) \right]$$

$$C_{max} = 1 + e^{-\frac{\xi \pi}{\sqrt{1-\xi^2}}}$$

$$-\frac{\xi \pi}{\sqrt{1-\xi^2}}$$

$$\%OS = e^{-\frac{\xi \pi}{\sqrt{1-\xi^2}}} \times 100$$

function of ξ only!!!

$$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

allows to solve for ξ
for a given %OS

→ settling time T_s

time it takes for the amplitude of the decaying sinusoid $= 0.02$

$$\frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} = 0.02$$

$$e^{-\xi\omega_n T_s} = 0.02 \sqrt{1-\xi^2}$$

$$-\xi\omega_n T_s = \ln(0.02 \sqrt{1-\xi^2})$$

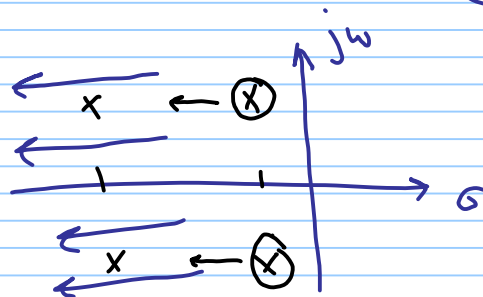
$$T_s = \frac{-\ln(0.02 \sqrt{1-\xi^2})}{\xi\omega_n}$$

$$\begin{array}{rcl} 0.001 & \leftarrow & \xi \\ 0.99 & \swarrow & \\ & & + 3.91 \\ & & \uparrow 7 \\ & & 5.8 \end{array}$$

$$T_s \approx \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d}$$

$$s_{1,2} = \sigma_d \pm j\omega_d$$

$$T_p = \frac{\pi}{\omega_d}$$



Rise Time T_r → from tables

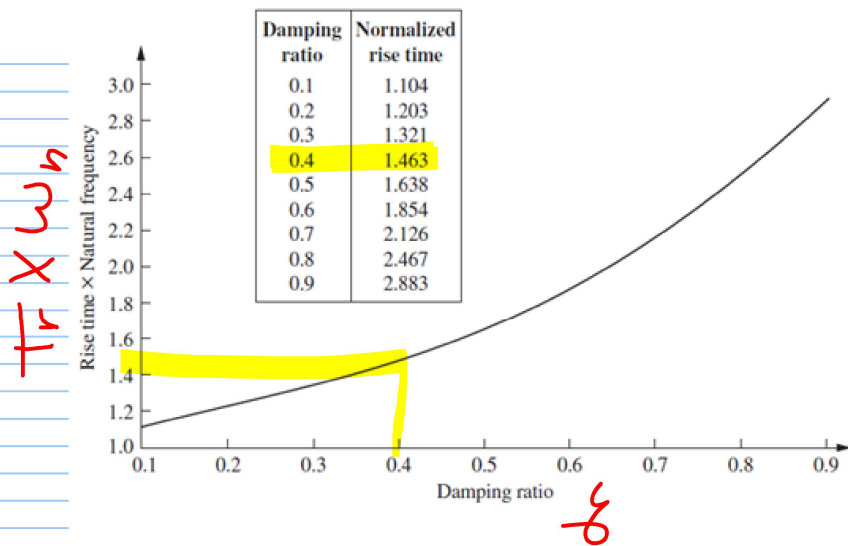


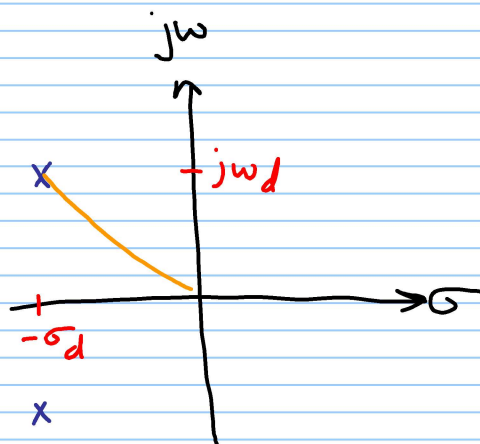
Figure: Normalized T_r vs. ζ for a 2nd-order underdamped response

$\zeta = 0.4$, so $T_r \times \omega_n = 1.463$

$T_r = \frac{1.463}{\omega_n}$ (when $\zeta = 0.4$)

Location of poles

$$\zeta = \frac{\sigma_d}{\omega_n}$$



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma_d \pm j\omega_d$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$T_r = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

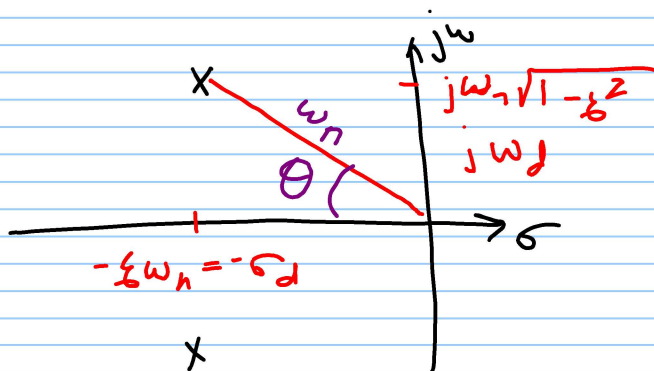
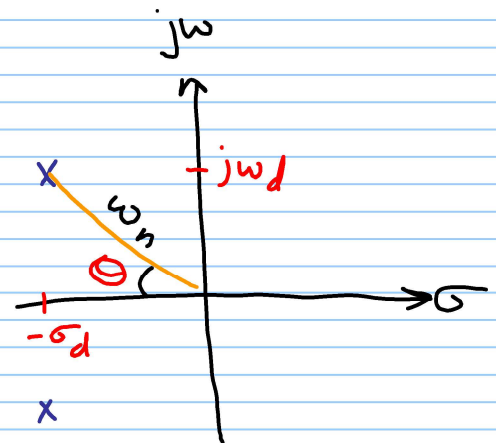
, T_r tables

$$\sqrt{\sigma_d^2 + \omega_d^2}$$

$$\sqrt{(\xi \omega_n)^2 + (\omega_n \sqrt{1-\xi^2})^2}$$

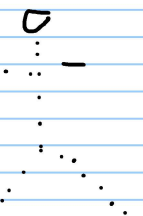
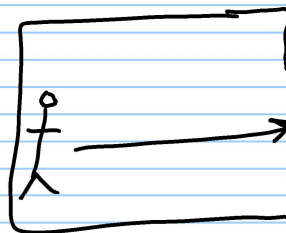
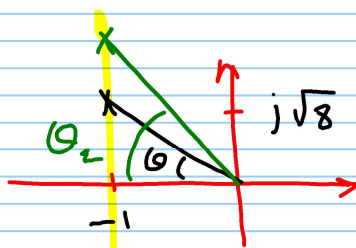
$$\sqrt{\xi^2 \omega_n^2 + \omega_n^2 (1-\xi^2)}$$

$$\sqrt{\cancel{\xi^2 \omega_n^2} + \omega_n^2 - \cancel{\omega_n^2 \xi^2}} = \omega_n$$



$$\xi = \cos \theta$$

$$\hookrightarrow \cos \theta = \frac{1 - \xi \omega_n}{\omega_n} = \xi$$

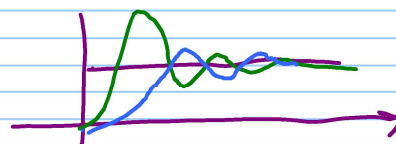


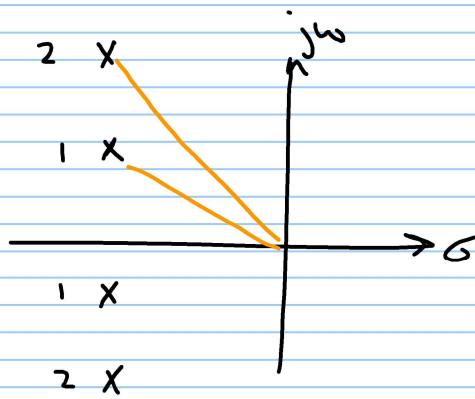
$$\xi = \cos \theta = \frac{1}{\omega_n} = \frac{1}{\sqrt{1+8}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

$$\theta_2 > \theta_1$$

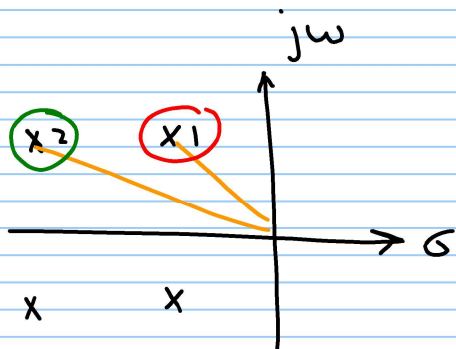
$$\cos \theta_2 < \cos \theta_1$$

$$\xi_2 < \xi_1 \Rightarrow \%OS_2 > \%OS_1$$





$$\begin{aligned} &\hookrightarrow \sigma_{d1} = \sigma_{d2} \\ &\therefore T_{F1} = T_{F2} \\ &\hookrightarrow \omega_{d1} < \omega_{d2} \\ &T_{P1} > T_{P2} \\ &\hookrightarrow \theta_1 < \theta_2 \\ &\cos \theta_1 > \cos \theta_2 \\ &\xi_1 > \xi_2 \\ &\%OS_1 < \%OS_2 \end{aligned}$$



$$\hookrightarrow \omega_{d1} = \omega_{d2}$$

$$T_{P1} = T_{P2}$$

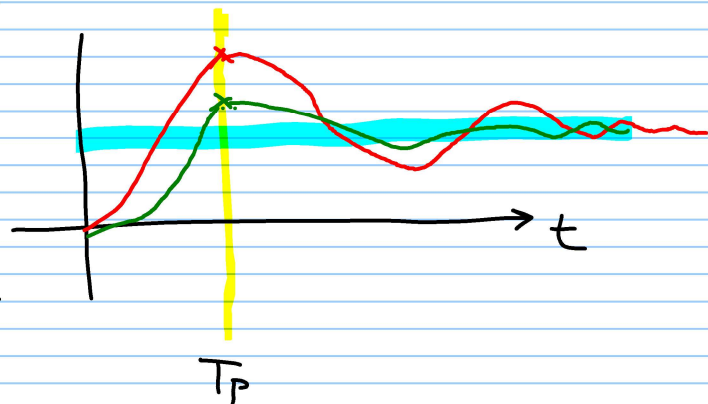
$$\hookrightarrow \sigma_{d1} < \sigma_{d2}$$

$$T_{F1} > T_{F2}$$

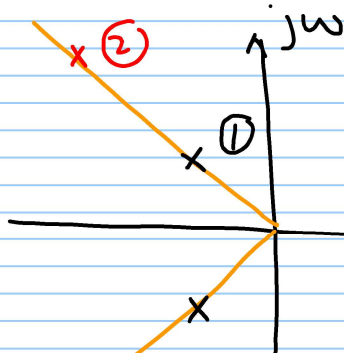
$$\hookrightarrow \theta_1 > \theta_2$$

$$\xi_1 < \xi_2$$

$$\underline{\%OS_1} > \underline{\%OS_2}$$



$$\xi = \cos \theta$$

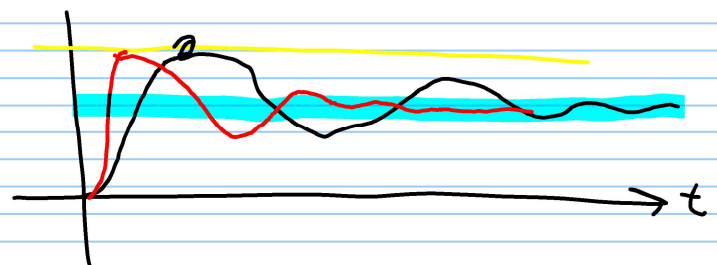


$$\hookrightarrow \omega_{d1} < \omega_{d2}$$

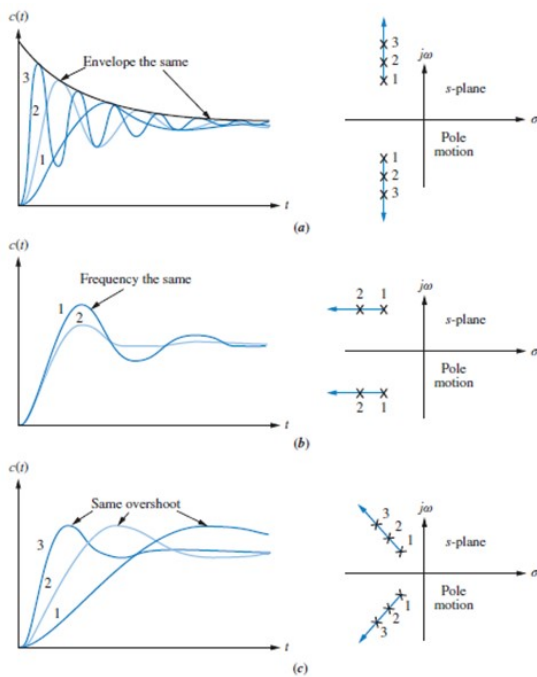
$$T_{P1} > T_{P2}$$

$$\hookrightarrow \sigma_{d1} < \sigma_{d2}$$

$$T_{F1} > T_{F2}$$



$$\theta_1 = \theta_2 \rightarrow \xi_1 = \xi_2 \rightarrow \%OS_1 = \%OS_2$$



Finding T_p , %OS, and T_s from Pole Location

PROBLEM: Given the pole plot shown in Figure 4.20, find ζ , ω_n , T_p , %OS, and T_s .

$$\sigma_d = 3$$

$$\omega_d = 7$$

$$\zeta = \cos \theta = \cos(\tan^{-1} \frac{7}{3})$$

$$= 0.394$$

$$\omega_n = \sqrt{3^2 + 7^2} = 7.616$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ sec.}$$

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} \text{ sec.}$$

$$\%OS = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100 = 26\%$$

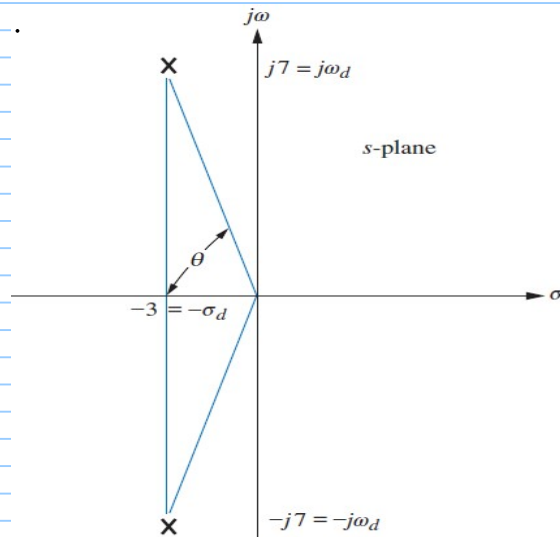


FIGURE 4.20 Pole plot for Example 4.6

Example 4.7

Transient Response Through Component Design

Design
D

PROBLEM: Given the system shown in Figure 4.21, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque $T(t)$.

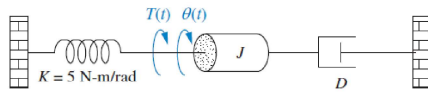


FIGURE 4.21 Rotational mechanical system for Example 4.7

$$(J s^2 + D s + 5) \theta(s) = T(s)$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$G(s) = \frac{1}{J s^2 + D s + 5}$$

$$\omega_n = \sqrt{5} \quad \times$$

$$= \frac{1/J}{s^2 + \frac{D}{J}s + \frac{5}{J}}$$

$$\omega_n = \sqrt{\frac{5}{J}}, \quad 2\zeta\omega_n = \frac{D}{J}$$

$$\rightarrow T_s = 2 = \frac{4}{\zeta\omega_n} \Rightarrow \zeta\omega_n = 2$$

$$\frac{D}{J} = 4$$

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

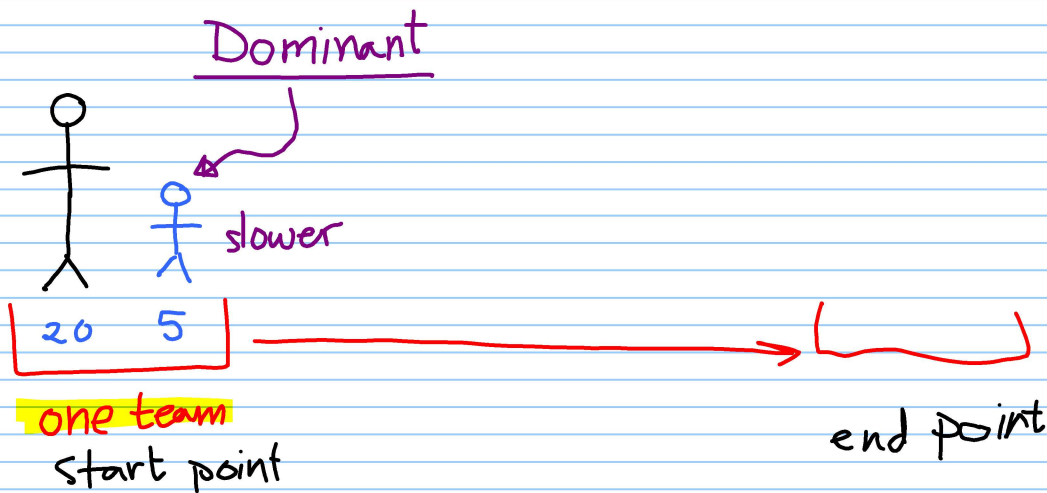
$$20 = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\ln\left(\frac{20}{100}\right) = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

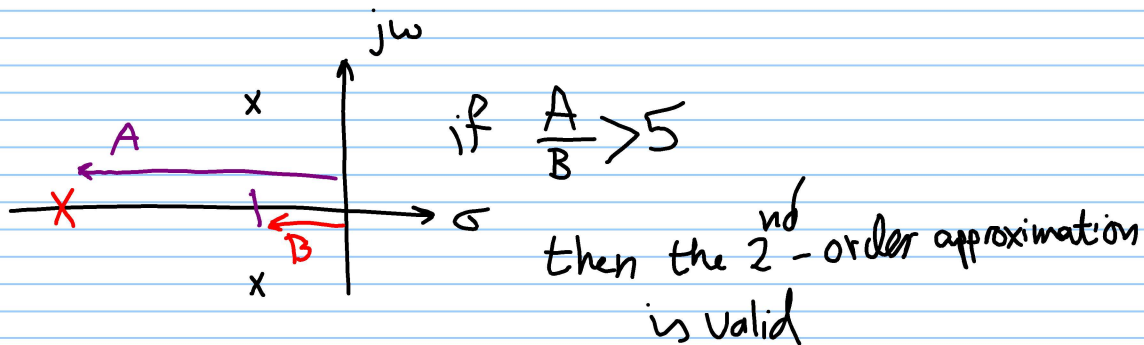
$$-1.61 = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = 0.456$$

$$\rightarrow \zeta\omega_n = 2 \quad \therefore \omega_n = \frac{2}{\zeta} = 4.386 \text{ rad/sec.}$$

$$\rightarrow \omega_n = \sqrt{\frac{5}{J}} \quad J = 0.26 \text{ kg}\cdot\text{m}^2 \quad \& \quad D = 4J = 1.04 \text{ N}\cdot\text{m/rad}$$



4.7 System response with additional poles



Example 4.8

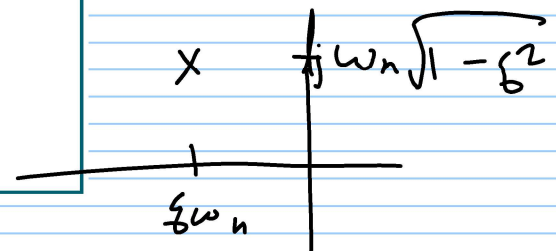
Comparing Responses of Three-Pole Systems

PROBLEM: Find the step response of each of the transfer functions shown in Eqs. (4.62) through (4.64) and compare them.

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542} \quad (4.62)$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.542)} \quad \checkmark \quad (4.63)$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.542)} \quad \times \quad (4.64)$$



$$T_2 / \frac{A}{B} = \frac{10}{2} = 5$$

$$T_3 / \frac{A}{B} = \frac{3}{2} = 1.5$$