Examples:

$$X(z) = \frac{1}{1-az^{-1}} \qquad |z| > a$$

$$x(z) = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

$$= \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

$$X(z) = \cancel{\sum}_{n=0}^{\infty} a^n z^{-n}$$

$$\Rightarrow x[n] = a^n u[n]$$

$$1 + az^{-1} + a^2 z^{-2}$$

$$1 - az^{-1} \,\big)\, 1$$

$$-\; 1 - az^{-1}$$

$$az^{-1}$$

$$-\; az^{-1} - a^2 z^{-2}$$

$$a^2 z^{-2}$$

$$\vdots$$

4- Contour Integration:

$$X[n] = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

Complex. integration

* Z- transform proberties

① Linearity

$$X[n] \xleftrightarrow{z} X(z) \;,\; ROC = R_x$$

$$X_1[n] \xleftrightarrow{z} X_1(z) \;,\; ROC = R_{x_1}$$

$$X_2[n] \xleftrightarrow{z} X_2(z) \;,\; ROC = R_{x_2}$$

$$a\,X_1[n] + b\,X_2[n] \xleftrightarrow{z} a\,X_1(z) + b\,X_2(z)$$

$$ROC = R_{x_1} \cap R_{x_2}$$

2- Time shifting

$$x[n] \xleftrightarrow{z} X(z)$$

$$x[n-n_0] \longleftrightarrow X(z)\, z^{-n_0}$$

ROC : Rx (except for possible addition or deletion of $z=0$ or $z=\infty$)

Example: let $X(z) = \frac{1}{z - 1/4}$, $|z| > 1/4$

from ROC $[|z| > 1/4]$ $x[n]$ is right-sided sequence

$$X(z) = \frac{z^{-1}}{1 - 1/4\, z^{-1}} \qquad |z| > 1/4$$

using the properties

$$X(z) = z^{-1}\left(\frac{1}{1 - 1/4\, z^{-1}}\right) \longleftrightarrow \overset{n_0 = -1}{x[n-n_0]}$$

$$x[n] = (1/4)^{n-1}\, u[n-1]$$

OR

$$-1/4\, z^{-1} + 1 \overline{\big)\, z^{-1}}^{\;-4}$$

$$-z^{-} - 4$$

$$4$$

$$\text{so } X(z) = -4 + \frac{4}{1 - 1/4\, z^{-1}}$$

$$x[n] = -4\,\delta[n] + 4\,(1/4)^n\, u[n]$$

Example :- Suppose $X(z)$ is given in the form

$$X(z) = z^2(1-\tfrac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$$

Evaluate $x[n]$

$$X(z) = z^2(1-1/2\,z^{-1})(1+z^{-1})(1-z^{-1})$$

$$= z^2(1-1/2\,z^{-1})(1-z^{-2})$$

$$= z^2(1-z^{-2}-1/2\,z^{-1}+1/2\,z^{-3})$$

$$= z^2 - 1/2\,z - 1 + 1/2\,z^{-1}$$

using shifting property

$$x[n] = 1\,\delta[n+2] - 1/2\,\delta[n+1] - \delta[n] + 1/2\,\delta[n-1]$$

3- Multiplying be exponential sequence

$$x[n] \xleftrightarrow{z} X(z)$$

$$z_0^n\, x[n] \longleftrightarrow \sum_{n=-\infty}^{\infty} z_0^n\, x[n]\, z^{-n}$$

$$= \sum x[n] \left(\frac{z}{z_0}\right)^{-n}$$

$$= X\left(\frac{z}{z_0}\right)$$

Example :- Consider the following sequence

$$x[n] = r^n \cos(w_0 n)\, u[n]$$

Evaluate $X(z)$

$$x[n] = r^n \cos(w_0 n)\, u[n]$$

$$= \underbrace{\tfrac{1}{2} r^n e^{jw_0 n}}_{\underset{a}{(re^{jw_0})^n}} u[n] + \tfrac{1}{2} r^n e^{-jw_0 n} u[n]$$

$$= \frac{1}{2} \frac{1}{1 - re^{jw_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - re^{-jw_0} z^{-1}}$$

4- Differentiation of $X(z)$

$$x[n] \longleftrightarrow X(z)$$

$$n\,x[n] \longleftrightarrow -z \frac{dX(z)}{dz} \quad , \text{ROC} = R_x$$

Example $X(z) = \log(1 + az^{-1}) \quad , |z| > |a|$

Evaluate $x[n]$

$$\frac{dX(z)}{d\cancel{z}} = \frac{-1(az^{-2})}{1 + az^{-1}}$$

$$n\,x[n] \overset{z}{\longleftrightarrow} -z \frac{dX(z)}{dz} = -z\left(\frac{-az^{-2}}{1 + az^{-1}}\right)$$

$$= \frac{az^{-1}}{1 + az^{-1}}$$

linearity and shift property

$$n\,x[n] = a(-a)^{n-1}\,u[n-1]$$

$$x[n] = \frac{a(-a)^{n-1}\,u[n-1]}{n}$$

$$= (-1)^{n-1}(a)^{n}\,u[n-1]$$

Example: $x[n] = n\,a^{n}\,u[n]$, evaluate $X(z)$

$$x[n] = n\,a^{n}\,u[n]$$

$$X(z) = -z\,\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) \qquad |z| > |a|$$

$$= -z\,\frac{(-az^{-2})}{(1-az^{-1})^{2}}$$

$$= \frac{a\,z^{-1}}{(1-az^{-1})^{2}}$$

$$n\,a^{n}u[n] \xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^{2}}$$

5- Conjugate of a Complex sequence:-

$$X[n] \xleftrightarrow{z} X[z] \qquad ROC = R_x$$

$$X^*[n] \longleftrightarrow X^*(z^*) \qquad RoC = R_x$$

6- Time Reversal:-

$$X[n] \xleftrightarrow{z} X[z]$$

$$X[-n] \longleftrightarrow X^*(1/z^*)$$

$$X(z^{-1})$$

$$RoC = \frac{1}{Rx} \text{ and inverted}$$

Example:- $X(n) = a^{-n} u[-n]$

Evaluate $X(z)$

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad RoC : R_x$$

$$a^{-n} u[-n] \longleftrightarrow \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$$

$$|z| < a^{-1} \qquad RoC = \frac{1}{R_x} = R_x^{-1}$$

$$1/a$$

7- Convolution sequence

$$X_1[n] * x_2[2] \xleftrightarrow{z} X_1(z)\ X_2(z)$$

RoC contains $R_{x1} \cap R_{x2}$

$$y[n] = \sum_{k=-\infty}^{\infty} X_1(k)\, X_2(n-k)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]\, z^{-n}$$

$$= \sum_{k=\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_1(k)\, X_2(n-k)\ z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} X_2(n-k)\, z^{-n}$$

let $m = n-k$ at $n=-\infty$ $m=-\infty$

$n = \infty$ $m = \infty$

$$\sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m)\ z^{-(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{m=-\infty}^{\infty} x_2(m)\ z^{-m}\right] z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{m=-\infty}^{\infty} x_2(m)\, z^{-m}$$

$$= X_1(z)\ X_2(z)$$

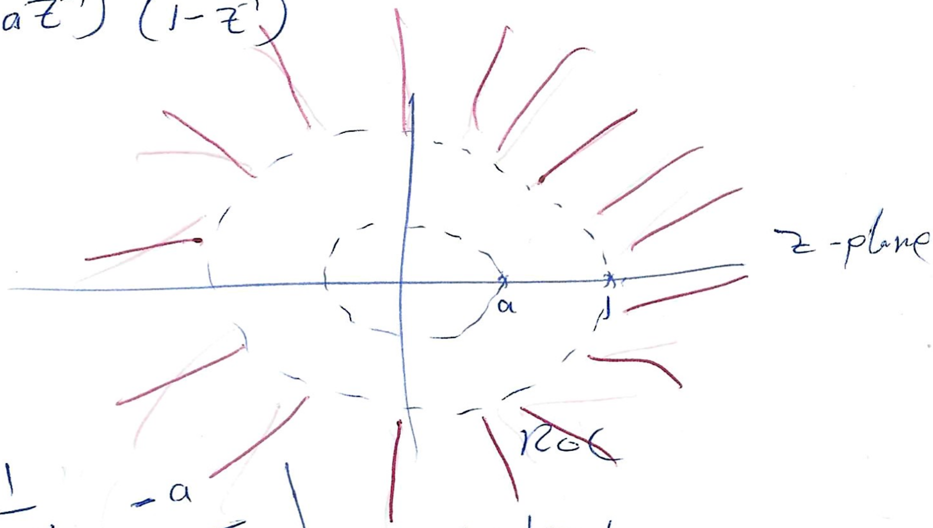
Example: $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$

Evaluate $y[n] = x_1[n] * x_2[n]$

$$X_1(z) = \frac{1}{1 - az^{-1}} \qquad |z| > a$$

$$X_2(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} \qquad |z| > 1$$



$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1 - z^{-1}} \quad \frac{-a}{1 - az^{-1}}\right) \qquad |z| > 1$$

$$\therefore y[n] = \frac{1}{1-a}\left(u[n] - a\, a^n u[n]\right.$$

$$= \left(\frac{1}{1-a}\right)[un \quad -a^{n+1}u[n]]$$

8- Initial value theorem

$x[n] \longleftrightarrow X[z]$ and $x[n]$ Causal

$$x[0] = \lim_{z \to \infty} X(z)$$