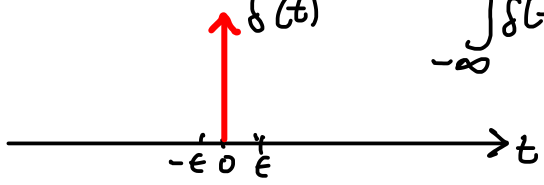
Test Signals used in Control Systems

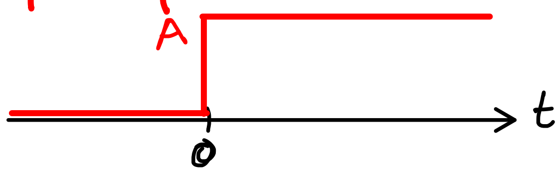
Unit-Impulse function:



$$\int_{-\infty}^{+\infty} \delta(t)\,dt = \int_{-\epsilon}^{+\epsilon} \delta(t)\,dt = 1$$

$$\mathcal{L}\{\delta(t)\} = 1$$

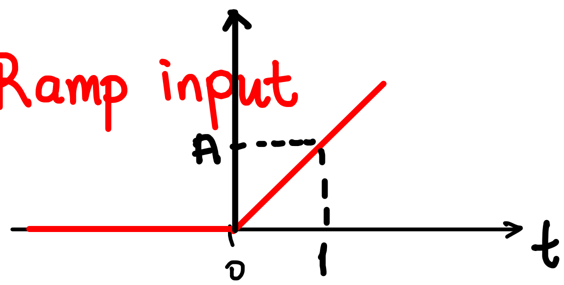
Step-Input:



in literature mostly denoted by $r(t)$, $u_c(t)$, $u_s(t)$, $u(t)$

$$r(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad R(s) = \frac{A}{s}$$

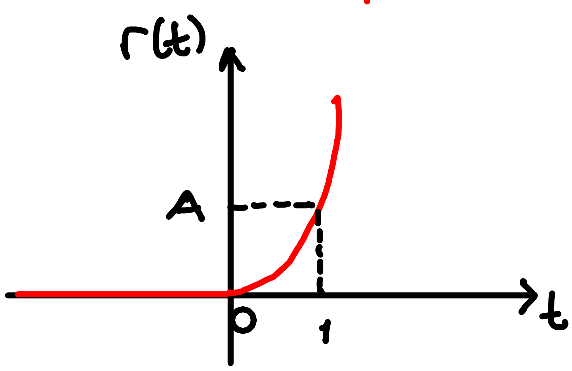
Ramp input



$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$R(s) = \frac{A}{s^2}$$

Parabolic input:



$$r(t) = \begin{cases} At^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

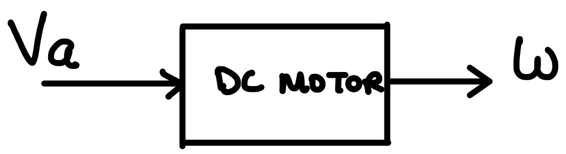
$$\mathcal{L} \downarrow$$

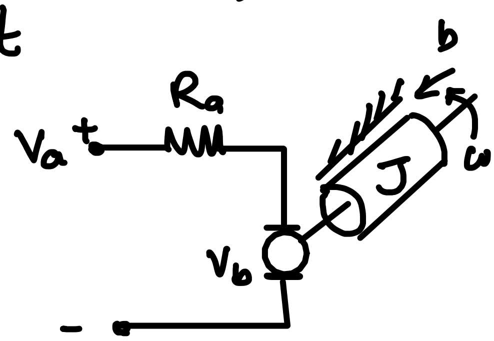
$$R(s) = \frac{2A}{s^3}$$

In general if $r(t) = t^n$, $t \geq 0$

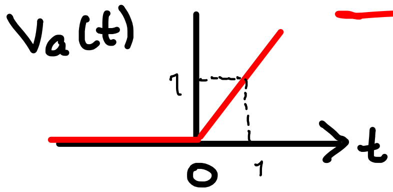
$$\Rightarrow R(s) = \frac{n!}{s^{n+1}}$$

EX: Let us calculate the unit ramp response of the open-loop DC-motor where $J = b = 1$, $K_b = K_m = R_a = 1$ and $V_a(t) = t$





$$\left.\frac{\omega(s)}{V_a(s)}\right|_{T_d=0} = \frac{K_m}{R_a b + K_b K_m + Js} = \frac{1}{s+2}$$



$\rightarrow V_a(s) = \frac{1}{s^2}$

$$\omega(s) = \frac{1}{s+2} \cdot \frac{1}{s^2} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2}$$

$$A = 1/2 \quad , \quad C = 1/4$$

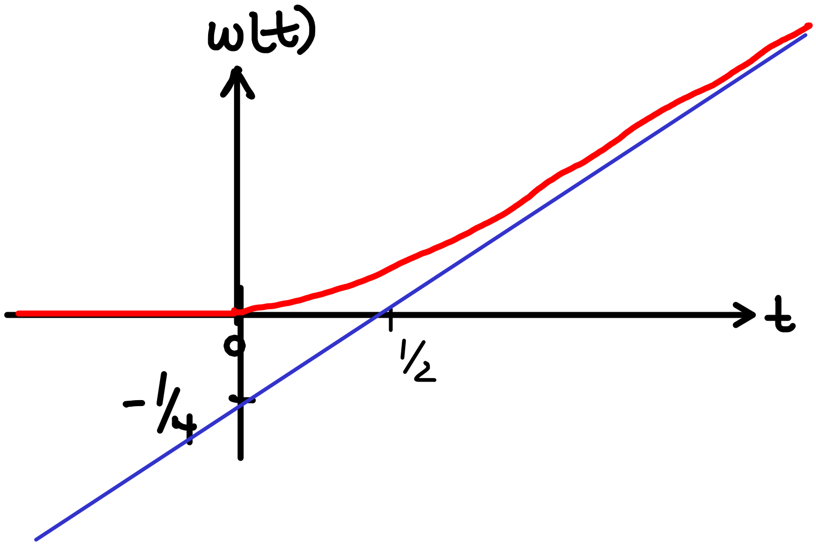
$$\frac{1}{3} \overset{!}{=} \frac{1/2}{1} + \frac{B}{1} + \frac{1/4}{3}$$

$$\Rightarrow B = -\frac{1}{4}$$

$$\omega(s) = \frac{1/2}{s^2} + \frac{-1/4}{s} + \frac{1/4}{s+2}$$

$\downarrow \mathcal{L}^{-1}$

$$\omega(t) = \left[\frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t}\right]u(t)$$

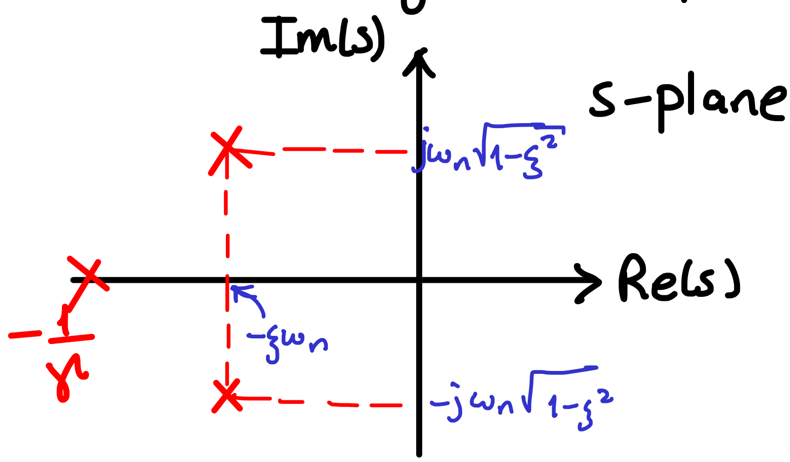


3rd order systems

$$T(s) = \frac{\omega_n^2}{(s^2+2\xi\omega_n s+\omega_n^2)(\gamma s+1)}$$

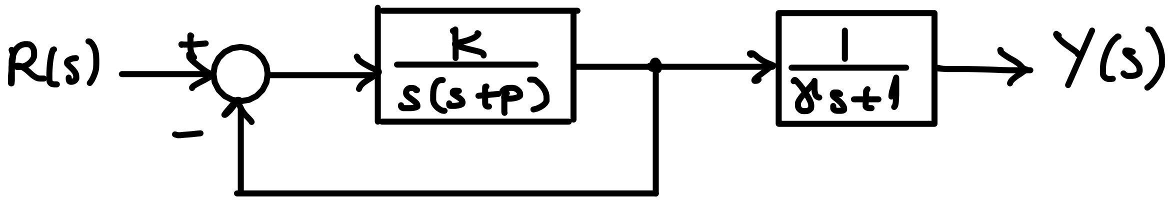
↑ add a pole @ $-\frac{1}{\gamma}$

if $-\frac{1}{\gamma} < -\xi\omega_n$, then the pole-zero configuration of $T(s)$ is as shown:



NOTE: if $\boxed{\frac{1}{\gamma} \geq 10\,\xi\omega_n}$ the unit-step response of the 3rd-order system is very similar to that of a 2nd order system. In such a situation, the poles $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ are called "dominant poles" and the pole at $-\frac{1}{\gamma}$ is called an insignificant pole. If $\frac{1}{\gamma} \gg 10\xi\omega_n$ then the response of the 3^{rd} order system is significantly different from that of the 2^{nd} order system. Then we can't use the expressions derived for T_s, T_p, T_r, etc.

EX: Consider the fb system shown below



Select K, p, γ such that the unit-step response is closed to a 2^{nd}-order system unit step response with an overshoot of less than 5% and $T_s \leq 4$ secs.

Note that
$$\frac{Y(s)}{R(s)} =: T(s) = \frac{\frac{1}{\gamma s+1} \cdot \frac{K}{s(s+p)}}{1+\frac{K}{s(s+p)}}$$

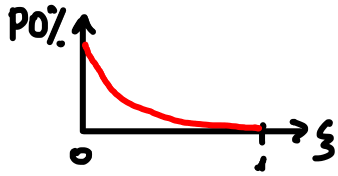
$$T(s) = \frac{K}{(s^2+ps+K)(\gamma s+1)} \overset{!}{\underset{\sim}{=}} \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Since we want $T_s \leq 4$ secs, $T_s = \frac{4}{\xi\omega_n} \leq 4$

$\Rightarrow \boxed{\xi\omega_n \geq 1}$ (1) From the overshoot equation

$$\xi = \frac{-\ln PO\%}{\sqrt{\pi^2 + \ln^2 PO\%}}$$

we know that for an overshoot of less than 5% , $\boxed{\xi \geq 0.7}$ (2)

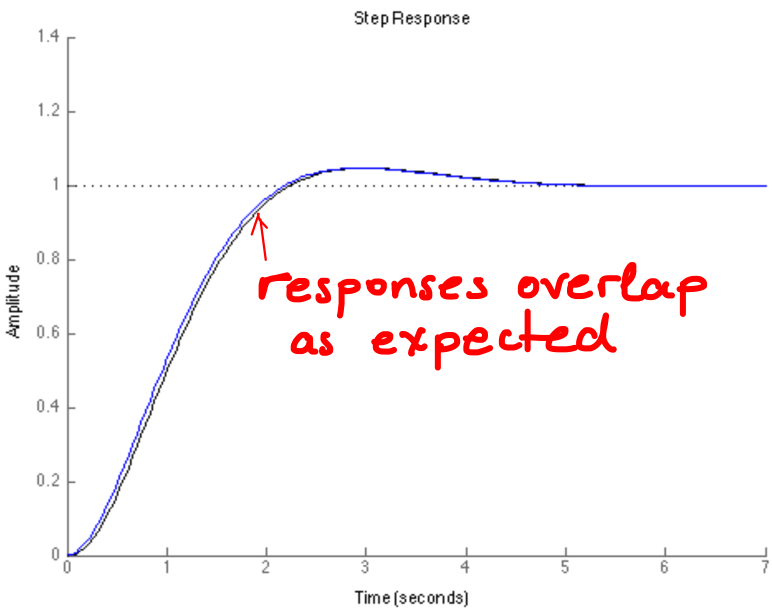


so setting $\underline{\xi = 0.7}$, and $\underline{\omega_n = 1.5}$ will satisfy (1).

$$\Rightarrow \quad \frac{1}{\gamma} \geq 10\,\xi\,\omega_n = 10 \times 0.7 \times 1.5 = 10.5$$

$$0{,}095 = \frac{1}{10.5} \geq \gamma \quad \Rightarrow \text{choose } \gamma = 0.05$$

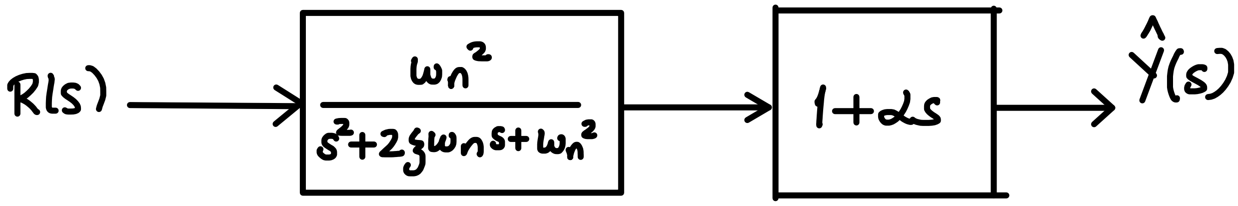
$$2\xi\omega_n \stackrel{!}{=} p \Rightarrow p = 2.1 \qquad K = \omega_n^2 \Rightarrow K = 2{,}25$$




```
K=2.25;
p=2.1;
gamma=0.05;

G3 = tf(K,conv([1 p K], [gamma 1]));
G2 = tf(K,[1 p K]);
step(G3,'k')
hold
step(G2, 'b')
box off
```

Adding a Zero to the t.f of a 2nd order System



Note that

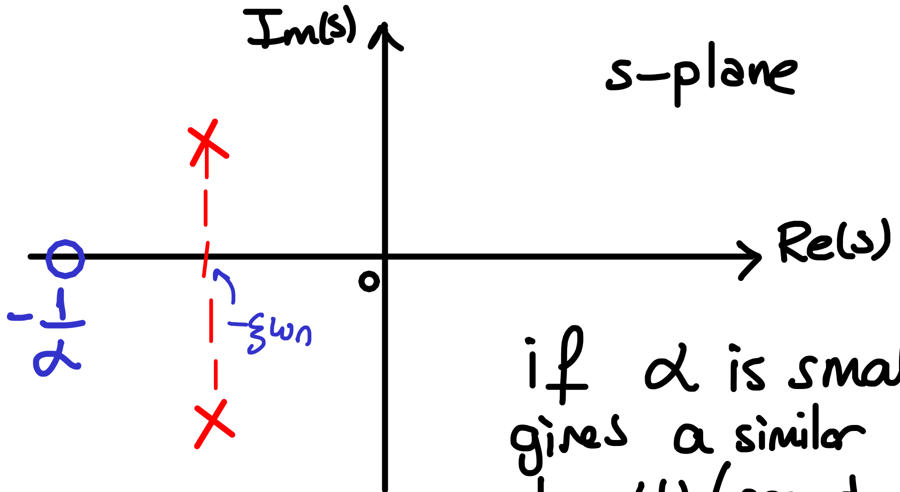
$$\frac{\hat{Y}(s)}{R(s)} = \frac{\omega_n^2(1+\alpha s)}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} + s\,\frac{\alpha\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\hat{Y}(s) = \boxed{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s)} + \boxed{s\alpha\,\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s)}$$

$$\hat{Y}(s) = Y(s) + s\alpha Y(s)$$

$$\hat{y}(t) = \left(\text{response of a second order prototype syst.}\right) + \alpha \times \left(\text{derivative of the second order prototype syst. response}\right)$$

$$\Rightarrow \hat{y}(t) = y(t) + \alpha \frac{d}{dt} y(t)$$



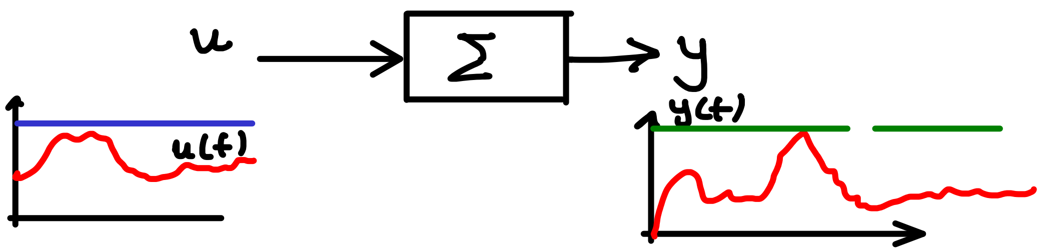
if α is small this gives a similar response to $y(t)$ (second order prototype syst. resp)

Again if $\frac{1}{\alpha} \geq 10\zeta\omega_n$, then we have a negligible zero and poles @ $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ are called 'dominant'.

Stability of Linear Systems

Definition (Bounded Input-Bounded Output Stability) A linear time-invariant system is BIBO stable if one of the following conditions hold:

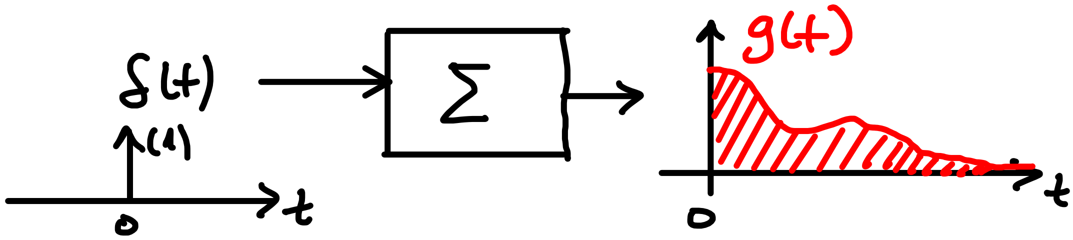
(1) Its output is bounded for every bounded input i.e., $|u(t)| < \infty \;\; \forall t \Rightarrow |y(t)| < \infty \;\; \forall t$. where u is the input, y is the output of the system



NOTE: Not a good way to test the stability!

(2) Its impulse response $g(t)$ is absolutely integrable i.e.

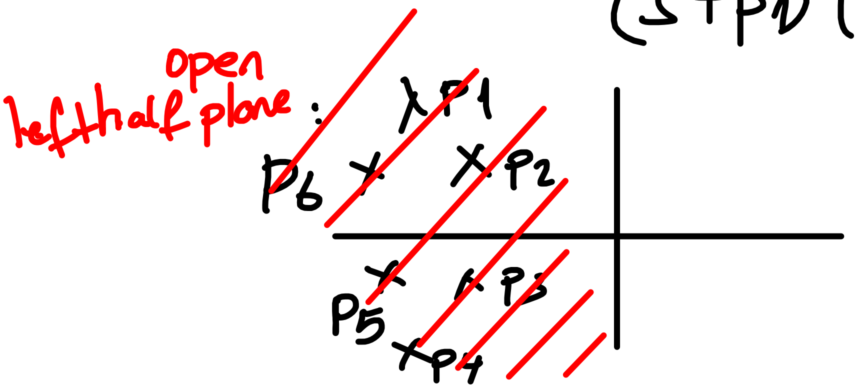
$$\int_{-\infty}^{+\infty} |g(t)|\,dt < \infty$$



(3) All poles of its transfer-function $G(s)$

$G(s) = \mathcal{L}\{g(t)\}$ are in the open left half plane

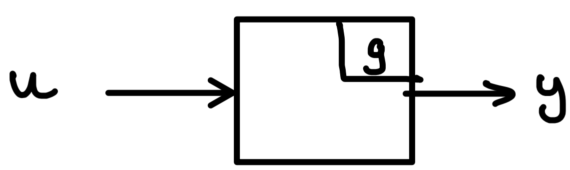
$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$$



proof (ii) $\Rightarrow$ (i)

Assume $g(t)$ is absolutely integrable.

$$\int_0^\infty |g(t)|\,dt < K < \infty \quad \text{for some } K>0$$



$Y(s) = G(s)U(s)$

$y(t) = \int_0^\infty g(\tau)\,u(t-\tau)\,d\tau$

$|A+B| \leq |A| + |B|$

$$|y(t)| = \left|\int_0^\infty g(\tau)u(t-\tau)\,d\tau\right| \leq \int_0^\infty |g(\tau)u(t-\tau)|\,d\tau$$

$$= \int_0^\infty |g(\tau)|\cdot|u(t-\tau)|\,d\tau$$

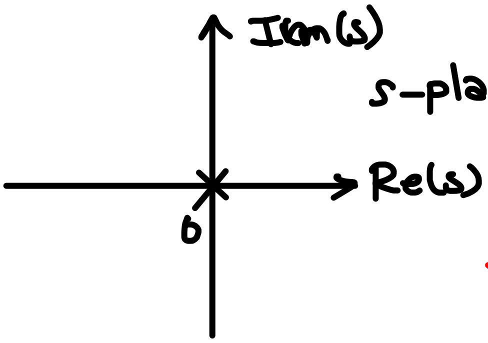
Let $|u(t-\tau)| < M < \infty$ for all $t \geq 0$

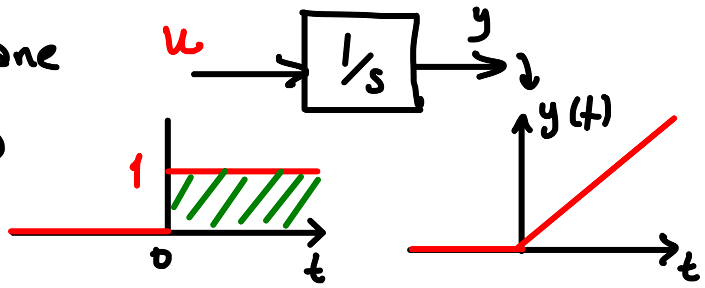
$$|y(t)| \leq \int_0^\infty |g(\tau)| \cdot \underbrace{|u(t-\tau)|}_{\leq M} d\tau \leq M \underbrace{\int_0^\infty |g(\tau)| d\tau}_{\leq K}, \ \forall t \geq 0$$

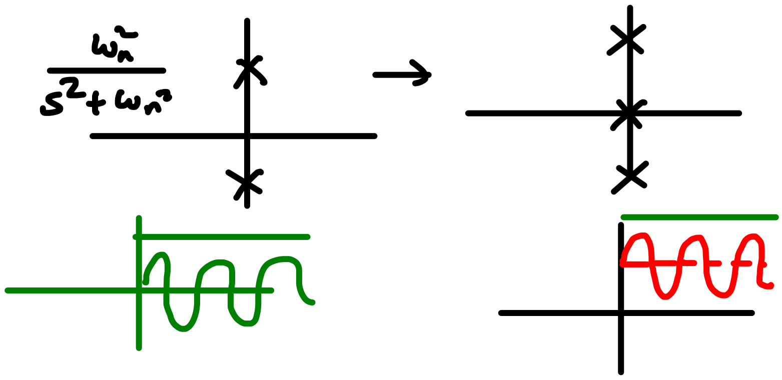
$$\Rightarrow |y(t)| \leq M \cdot K < \infty$$

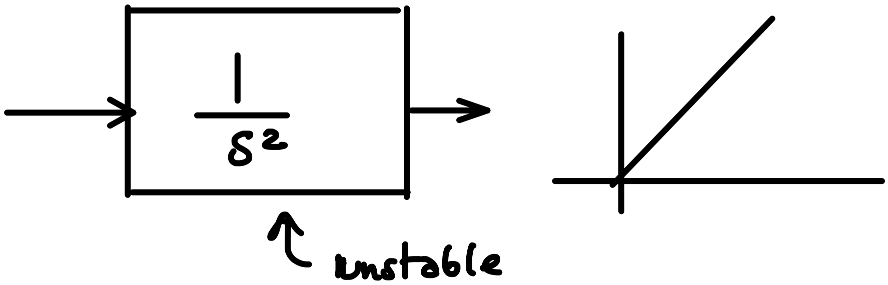
the rest of proofs are left as an exercise.

marginally stable

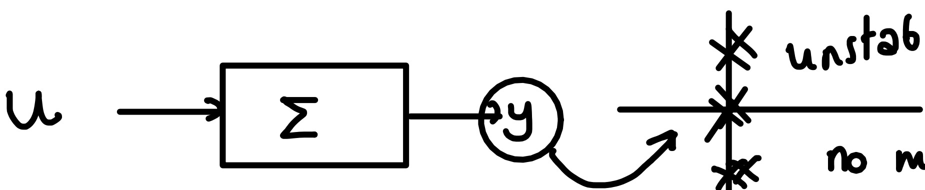








unstable



unstable

no multiple roots are allowed on imaginary axis

$Y(s) = \frac{1}{s^2(s+4)}$ $\quad$ $Y(s) = \frac{1}{s(s+4)}$

SUMMARY:

A system is:

1. stable if all poles are on the open left half plane
2. unstable if one of its poles is in the right half plane
3. marginally stable if it has simple poles on the imaginary axis.
4. unstable if it has repeated poles on the imaginary axis.

Routh-Hurwitz Criterion

Consider $\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$

$$\frac{Y(s)}{R(s)} = \frac{s^m + r_{m-1} s^{m-1} + \cdots r_0}{\Delta(s)}$$

$\Delta(s)$ ↑ characteristic poly.

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	$\cdots$
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
s^1	x_{n-1}			
s^0	z_{n-1}			

$$b_{n-1} = \frac{a_{n-1} \cdot a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-3} = \frac{a_{n-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_{n-1} = \frac{b_{n-1} a_{n-3} - a_{n-1} b_{n-3}}{b_{n-1}} \quad \cdots\cdots$$

Fact 1: $\Delta(s)$ has at least one root in the closed right half plane $\Leftrightarrow$ there are some zeros or sign changes in the first column of the array.

Fact 2: If there are no zero entries in the first column then the number of sign changes in the first column is equal to the number of roots in the RHP.

EX: $\Delta(s) = a_2 s^2 + a_1 s + a_0$

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 & \end{array}$$

$\Rightarrow$ No roots in the RHP iff a_2, a_1, a_0 have the same sign.

if $\exists$ zeros in the first column

Case 1

All elements of the row are zero

[construct an auxillary poly. from the previous row]

Case 2

The row contains a nonzero element

[Replace 0 with $\epsilon > 0$ and continue.]

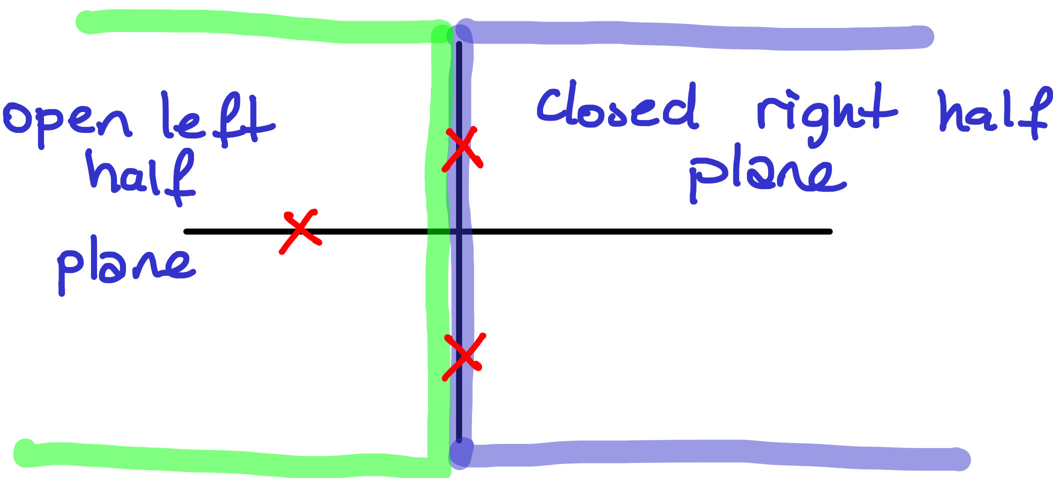
Ex: $\Delta(s) = s^3 + \underline{a}s^2 + \underline{b}s + \underline{a}b$

s^3	1	b
s^2	a	ab
s^1	$\cancel{0}$ $2a$	0
s^0	ab	

$\longrightarrow \Delta_a(s) = as^2 + ab$

$\Delta_a'(s) = \underline{2a}s$

if $a>0$, $b>0$ then roots of the auxiliary polynomial are on the RHP. Since there is no sign changes



then 2 poles must be on the imaginary axis.
∴ the system is marginally stable.

EX (case 2) Consider $\Delta(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

s^5	1	2	11
s^4	$\cancel{2}$ 1	$\cancel{4}$ 2	$\cancel{10}$ 5
s^3	$\cancel{0}$ ϵ	6	
s^2	$\frac{2\epsilon-6}{\epsilon}$	5	
s^1	6		
s^0	5		

sign changes in 1st column

+	+
+	+
−	+
+	−
+	+
+	+

2 sign changes

2 RHP poles ⟹ UNSTABLE

EX: $\Delta(s) = s^3 + as^2 + bs + (ab + \epsilon)$

s^3	1	b
s^2	a	$ab+\epsilon$
s^1	$-\epsilon/a$	
s^0	$ab+\epsilon$	

<u>if $\epsilon = 0$</u>

construct auxillary poly when $\epsilon = 0$

$\Delta_a(s) = as^2 + ab$
$= a(s^2 + b)$

$(s + j\sqrt{b})(s - j\sqrt{b})$

$\Rightarrow$ symmetrical poles w.r.t. origin

Q: what to do?

A: In case of (1), form an auxillary poly using the coefficients of the row before the zero row, replace the zero row coefficients with the coefficients of the derivative of the auxillary poly. and continue.

NOTE: if $j\omega$-axis roots of the characteristic polynomial are simple (not repeated) then the system is neither stable nor unstable and called "marginally stable". If the $j\omega$-axis roots are repeated, the system is unstable.

EX: $\Delta(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$

s^5	1	2	1
s^4	1	2	1
s^3	$\cancel{0}\ \cancel{4}\ 1$	$\cancel{0}\ \cancel{4}\ 1$	0
s^2	1	1	
s^1	$\cancel{0}\ 2$	0	
s^0	1		

$\rightarrow \Delta_{a_1}(s) = \boxed{s^4 + 2s^2 + 1}$

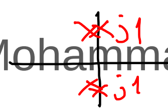
$\Delta'_{a_1}(s) = 4s^3 + 4s$

4 poles are at RHP.

$\rightarrow \Delta_{a_2}(s) = \boxed{s^2 + 1}$

$s^4 + 2s^2 + 1 = (s^2 + 1)^2$

$\Delta'_{a_2}(s) = 2s$



UNSTABLE