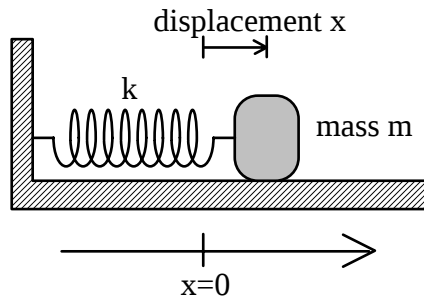


Lab M5: Hooke's Law and the Simple Harmonic Oscillator

Most springs obey Hooke's Law, which states that the force exerted by the spring is proportional to the extension or compression of the spring from its equilibrium length.

$$(1) \quad F = -kx.$$



k is called the spring constant and is a measure of the stiffness of the spring. The minus sign indicates that the direction of the force is opposite to the direction of the displacement. In the SI system, the units of spring constant k are N/m. The diagram shows a mass to the right of its equilibrium position ($x=0$). The displacement of the mass is (+) and the spring force is to the left (-). If the spring is compressed, then x is

(-) and the spring force is (+).

Simple harmonic motion occurs whenever there is a restoring force which is proportional to the displacement from equilibrium, as is the case here. Assuming no frictional forces and *assuming that the spring is massless*, the equation of motion ($ma = F$) of a mass on a spring is

$$(2) \quad m \frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

The solution of this second-order differential equation is

$$(3) \quad x(t) = A \sin(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}},$$

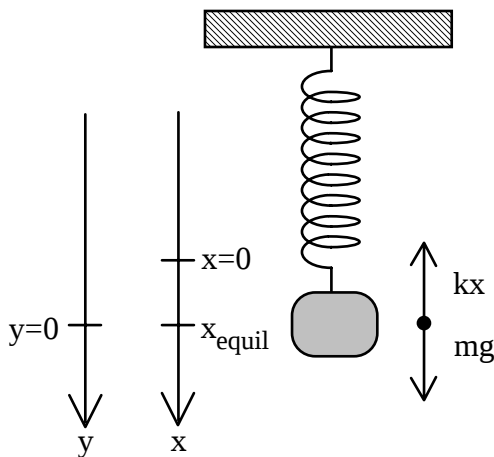
where A and ϕ are constants which depend on the initial conditions, the initial position and initial velocity of the mass. The period T , the frequency f , and the constant ω are related by

$$(4) \quad \omega = 2\pi f = \frac{2\pi}{T}.$$

Thus the period T is given by

$$(5) \quad T = 2\pi \sqrt{\frac{m}{k}}.$$

A very important property of simple harmonic motion is that the period T does not depend on the amplitude of the motion, A .



If the mass is hung from a vertical spring, it will still execute simple harmonic motion with the same period (5), as we will now show. When the mass is hung from the spring, the spring is stretched from its equilibrium length by the weight mg of the mass. The equilibrium displacement of the mass under the influence of the force of gravity (down) and the force from the spring (up) is $x_{\text{equil}} = \frac{mg}{k}$.

Now that there are two forces acting on the mass, the equation of motion becomes

$$(6) \quad m \frac{d^2 x}{dt^2} = mg - kx.$$

We now perform a change of variable by introducing a new coordinate $y = x - \frac{mg}{k}$.

Since $\frac{mg}{k}$ is a constant, differentiating y twice produces $\frac{d^2 y}{dt^2} = \frac{d^2 x}{dt^2}$. Also, $mg - kx = -ky$, therefore eq'n (6) becomes

$$(7) \quad m \frac{d^2 y}{dt^2} = -ky,$$

which is exactly the same as eq'n (2) except we have changed the name of the position coordinate from x to y . Since the equation of motion is the same, the solution is the same (3), with the same period (5).

A real spring has mass, a fact which we have ignored so far. A mass m on a real spring with mass m_{spring} oscillates more slowly than predicted by (5), since the spring has to push its own mass about as well as the mass m . However, the theoretical expression (5) can be corrected by replacing the mass m with an *effective mass*, consisting of m plus some fraction of m_{spring} .

$$(8) \quad m_{\text{eff}} = m + f \cdot m_{\text{spring}},$$

where f is some fraction ($f < 1$), which depends on the exact shape of the spring. For the spring used in this lab, the fraction f has been determined experimentally to be

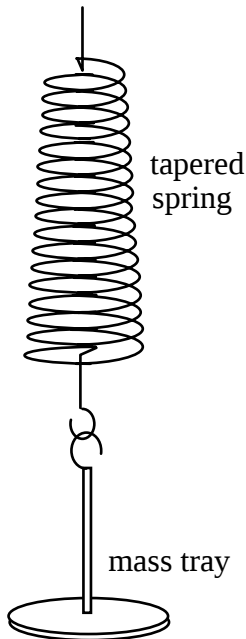
$$(9) \quad f = 0.281 \pm 0.002.$$

[Although f can be computed, the computation is rather complicated and depends on the precise shape of the spring.]

In this experiment, you will first determine the spring constant k of a spring, by hanging various weights from the spring and measuring the extension. Then, you will measure the period of oscillation of a mass m hanging from the spring, and will compare this measured period with period predicted by eq'n (5) [with m_{eff} used instead of m]. Finally, you will repeat your measurement of the period with two other masses and check that the period is proportional to $\sqrt{m_{\text{eff}}}$.

Procedure

Part 1: Measurement of the spring constant k



Begin by weighing the spring (you will need the weight in part 2). You might want to check the reliability of the digital balances by weighing the spring on two different balances. Use the digital balances with 0.1g resolution. The spring used in this lab has a tapered coil which serves to reduce the interference of the windings with each other and make the spring behave more like a perfect Hooke's spring. Always hang the spring with the larger windings downward.

With the empty mass tray hanging from the spring, measure the position of an edge of the tray. This will be the zero position, which you will subtract from all subsequent positions. There is a mirror by the meter stick scale so you can avoid parallax when you measure positions. Now add the slotted weights to the tray, one at a time, and measure the positions of the same edge as each mass is added. Add masses in 50 gram increments to a total of 500 grams: $\Delta m = 50 \text{ g}, 100 \text{ g}, 150 \text{ g}, \dots$ Use the balance to check that the masses are accurate.

Plot the weight added ($\Delta m \cdot g$) vs. Δx , the position change from the zero position. Use the known value of $g = 9.796 \text{ m/s}^2$. This graph should be a straight line with slope k .

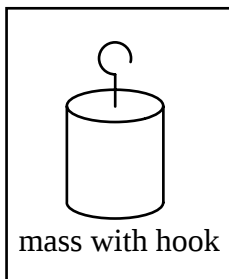
To determine the slope, compute $k = \frac{\Delta m \cdot g}{\Delta x}$ for each data point, and compute the mean, the standard deviation, and the standard deviation of the mean of your several k values.

$$k_{\text{avg}} = \frac{1}{N} \sum_{i=1}^N k_i \quad \sigma = \sqrt{\frac{\sum_{i=1}^N (k_i - k_{\text{avg}})^2}{N-1}} \quad \sigma_{\text{mean}} = \frac{\sigma}{\sqrt{N}}$$

[Side comment: You might think to determine the best value of k by measuring the slope of ($\Delta m \cdot g$) vs. Δx , using the “slope” function in Mathcad. This is not quite right

since what we want is the best fit line *which goes through the origin*, while the slope function finds the slope of the best fit line, which, in general, has a non-zero intercept.]

Part 2: Measurement of the period



Remove the weight tray from the spring and load the spring with a 100 gram mass with a hook. Carefully, set the mass oscillating with a small amplitude motion and use a stop watch to time the interval for several complete oscillations. If the time for N complete periods ($N \approx 50$ or more) is T_{total} , then the period is $T_{\text{meas}} = \frac{T_{\text{total}}}{N}$.

By measuring the time for many periods, the uncertainty in T , due to your reaction time, is reduced by a factor of N .

$$\delta T_{\text{total}} = \text{human reaction time} \approx 0.1 \text{ sec.} \quad \delta T = \frac{\delta T_{\text{total}}}{N}.$$

[DO NOT measure one period 50 times; measure the time for 50 periods, once.]

If you have time, repeat this measurement at least once, to check the reproducibility of this method.

Now compute the period using eq'n (5) with m_{eff} in place of m

$$(10) \quad T_{\text{calc}} = 2\pi \sqrt{\frac{m_{\text{eff}}}{k}}.$$

Also compute δT_{calc} , the uncertainty in T_{calc} (see note on uncertainty calcs, below). Compare the calculated and measured periods.

Repeat this procedure with $m = 200\text{g}$ and $m = 500\text{g}$ (but you need not repeat the calculation of δT_{calc} - it's too time-consuming.). Make a plot of T_{meas} vs. $2\pi \sqrt{\frac{m_{\text{eff}}}{k}}$, using your three data points. On the same graph, plot the line $y = x$. Finally, compute the quantity $R = \frac{T_{\text{meas}}}{T_{\text{calc}}}$ for each of your three data points. Do theory and experiment agree?

[Review: For a function like $f(x, y, z) = \frac{x^n y^m}{z^p}$, involving multiplication, division, and

powers, the fractional uncertainty is given by $\frac{\delta f}{f} = \sqrt{\left(n \frac{\delta x}{x}\right)^2 + \left(m \frac{\delta y}{y}\right)^2 + \left(p \frac{\delta z}{z}\right)^2}$.]

PreLab Questions:

1. What condition or conditions are necessary for simple harmonic motion to occur?
2. What are the units of the spring constant k in SI units, which are also called MKS units (MKS = meter-kilogram-seconds)? What are the units in the cgs system (centimeter-gram-second)?
3. A given spring stretches 15 cm when it is loaded with a 250 gram weight. What is the spring constant of the spring?
4. Sketch the graph ($\Delta m \cdot g$) vs. Δx for a mass on a spring. Indicate the slope and intercept.
5. What happens to the period of a mass-on-a-spring simple harmonic oscillator if the mass is doubled? What happens to the period if the spring constant is reduced by a factor of 3? How does the period depend on the amplitude of the oscillation?
6. True or False: for a simple harmonic oscillator consisting of a mass on a spring, the period measured when the mass is hanging vertically (figure on p.2) is different than the period measured when the spring and mass are supported horizontally (figure on p.1) .
7. Verify that eq'n (3) is a solution of the differential equation (2). [Hint: There was a very similar problem in the Simple Pendulum Lab M1.]
8. Show with a sketch what a graph of T^2 vs. $\frac{m_{\text{eff}}}{k}$ looks like. What are the slope and intercept?
9. Assuming that k , m_{eff} , δk , and δm_{eff} are known, show how to compute δT_{calc} , where T_{calc} is given in eq'n(10).
10. A mass m hanging from a spring with spring constant k is taken to the Mars, where g is $1/3$ of its value on Earth, and the period of the oscillation is measured. Is the measured period on the Mars the same as the period measured on the Earth? Explain your answer.