

(1)

The Gaussian solution or Poisson solution of the following heat problem

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, t > 0 \quad (1) \\ u(x, 0) = f(x), & -\infty < x < \infty, \quad (2) \end{cases}$$

is given by

$$u(x, t) = \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(x-y) e^{-\frac{y^2}{4kt}} dy$$

proof. Applying  $\mathcal{F.T}$  to both sides of (1) to get

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = k \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\frac{dU(\alpha, t)}{dt} = -k\alpha^2 U(\alpha, t)$$

STUDENTS-HUB.COM

or

$$\boxed{\frac{dU}{dt} + k\alpha^2 U = 0} \quad (3), \text{ where } U(\alpha, t) = \mathcal{F}\{u(x, t)\}$$

Next, Applying  $\mathcal{F.T}$  to the condition (2):

$$\mathcal{F}\{u(x, 0)\} = \mathcal{F}\{f(x)\}$$

$$\boxed{U(\alpha, 0) = F(\alpha)} \quad (4)$$

Uploaded By: anonymous

(2)

Now, Solve (3) with condition (4):

$$U(x,t) = A e^{-kx^2 t}$$

$$U(x,0) = A = F(x)$$

$$\Rightarrow U(x,t) = F(x) e^{-kx^2 t} \quad (5)$$

We know that  $(6) \quad \mathcal{F}\{e^{-\beta x^2}\} = \sqrt{\frac{\pi}{\beta}} e^{-\frac{x^2}{4\beta}}, \beta > 0$

See your notes

Here we choose,  $\frac{1}{4\beta} = kt$  or  $\boxed{\beta = \frac{1}{4kt}}$

in (6)  $\Rightarrow \mathcal{F}\{e^{-\frac{x^2}{4kt}}\} = \sqrt{4\pi kt} e^{-ktx^2}$

or  $e^{-kx^2 t} = \mathcal{F}\left\{\frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}}\right\} \quad (7)$

(7) in (5) gives

STUDENTS-HUB.com

Uploaded By: anonymous

$$U(x,t) = F(x) e^{-kx^2 t}$$

$$U(x,t) = \mathcal{F}\{f(x)\} \mathcal{F}\left\{\frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}}\right\}$$

$$u(x,t) = \mathcal{F}^{-1}\{U(x,t)\}$$

$$= \tilde{f}^{-1} \left\{ \tilde{f} \{ f(x) \} \tilde{f} \left\{ \frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}} \right\} \right\} \quad (3)$$

$$= f(x) * \frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}} \quad \text{(Convolution theorem)}$$

Constant in x

$$= \frac{1}{2\sqrt{\pi kt}} \left( f(x) * e^{-\frac{x^2}{4kt}} \right)$$

$$= \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} f(x-\eta) e^{-\frac{\eta^2}{4kt}} d\eta$$

□

Ex. Find the Gaussian solution of the problem

$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty, \quad t > 0 \\ u(x, 0) = e^{-x} \end{cases}$$

STUDENTS-HUB.com

Uploaded By: anonymous

Sol. Here we have  $k=1$ ,  $f(x) = e^{-x}$ .

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x+\eta} e^{-\frac{\eta^2}{4t}} d\eta$$

$$= \frac{e^{-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\eta^2}{4t} + \eta} d\eta \quad (4)$$

Now,  $-\frac{\eta^2}{4t} + \eta = -\frac{1}{4t} \left[ \eta^2 - 4t\eta + 4t^2 - 4t^2 \right]$

$$= -\frac{1}{4t} \left[ (\eta - 2t)^2 - 4t^2 \right]$$

$$= t - \left( \frac{\eta - 2t}{2\sqrt{t}} \right)^2$$

$$\therefore u(x, t) = \frac{e^{-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{t - \left( \frac{\eta - 2t}{2\sqrt{t}} \right)^2} d\eta$$

$$= \frac{e^{t-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left( \frac{\eta - 2t}{2\sqrt{t}} \right)^2} d\eta$$

STUDENTS-HUB.com

$$u = \frac{\eta - 2t}{2\sqrt{t}} \Rightarrow d\eta = 2\sqrt{t} du$$

Uploaded By: anonymous

$$= \frac{e^{t-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-u^2} \cdot 2\sqrt{t} du$$

$$= \frac{e^{t-x}}{2\sqrt{\pi t}} \cdot 2\sqrt{t} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{e^{t-x} \cdot 2\sqrt{t} \sqrt{\pi}}{2\sqrt{\pi t}} = e^{t-x}$$