Quadratic Functions : Parabolas

- The linear function $y = ax + b$ has degree one and it is a line
- The qudratic function $y = ax^2 + bx + c$ has degree two and it is parabola $(a \neq 0)$

Parabola

- → opens upward if $a > 0 \Rightarrow$ vertex is minimum
- → Opens downward if $a < 0 \Rightarrow$ vertex is maximum
- → Vertex is $(x, y) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ $\Delta = b^2 - 4ac$
- → Axis of symmetry is $x = -\frac{b}{2a}$
- → x-intercepts (or zeros of $f(x)$) : $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ if $\Delta \geq 0$
  - when $y = 0$
  - $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ if $\Delta \geq 0$
- → y-intercept is $y = c$
  - when $x = 0$

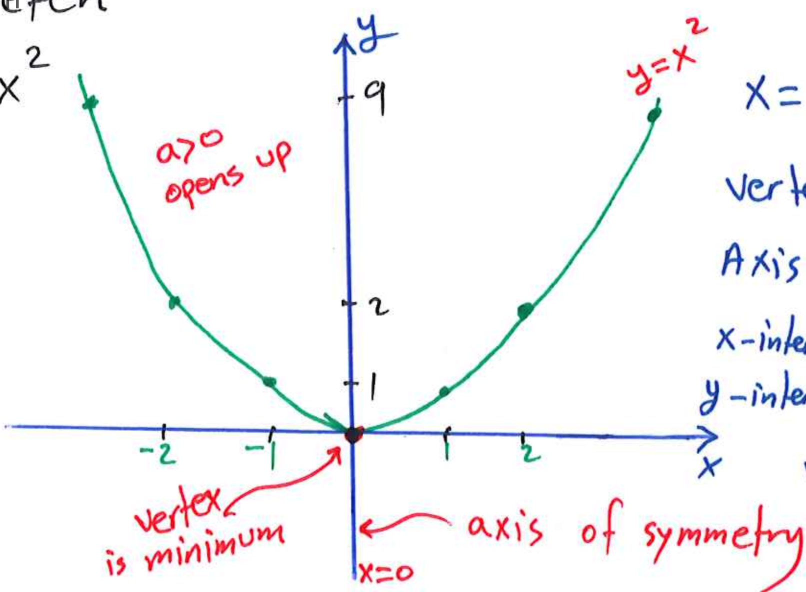
Remark : The optimum value (either maximum or minimum) occurs at the vertex point

---

Exp Find vertex, the optimum, axis of symmetry, intercepts and sketch

$\boxed{1}$ $f(x) = x^2$

x	$y = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



$a = 1, b = 0, c = 0$

$x = -\frac{b}{2a} = \frac{-0}{2} = 0$

Vertex is $(0, f(0)) = (0, 0)$

Axis of symmetry is $x = 0$

x-intercept $\Rightarrow y = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

y-intercept $\Rightarrow x = 0 \Rightarrow y = 0^2 = 0$

vertex is minimum since $a > 0$

$\boxed{2}$ $f(x) = -\frac{1}{2}x^2$

$a = -\frac{1}{2} < 0 \Rightarrow$ parabola opens downward

$\Rightarrow$ vertex is maximum

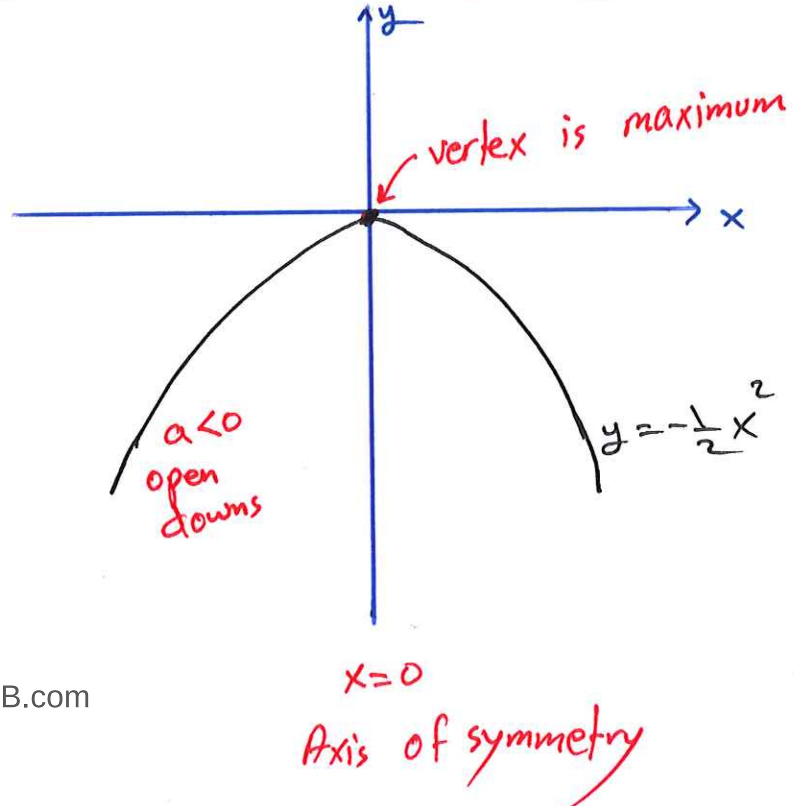
$b = 0,\ c = 0$ $\Rightarrow x = -\frac{b}{2a} = -\frac{0}{2(-\frac{1}{2})} = 0$

$\Rightarrow$ vertex is $\left(0, f(0)\right) = \left(0, -\frac{1}{2}(0)^2\right) = (0,0)$

$\Rightarrow$ Axis of symmetry is $x = -\frac{b}{2a} = 0$

x-intercept $\Rightarrow$ let $y=0 \Rightarrow -\frac{1}{2}x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

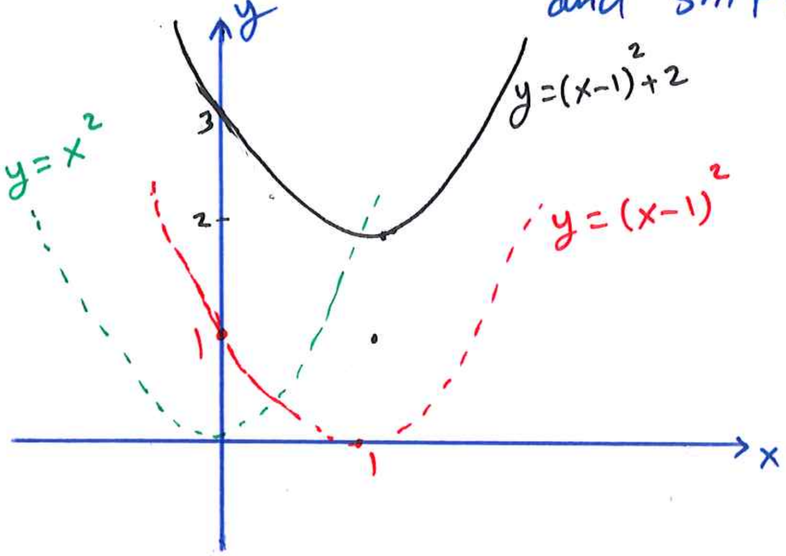
y-intercept $\Rightarrow$ let $x=0 \Rightarrow y = -\frac{1}{2}(0)^2 = 0$

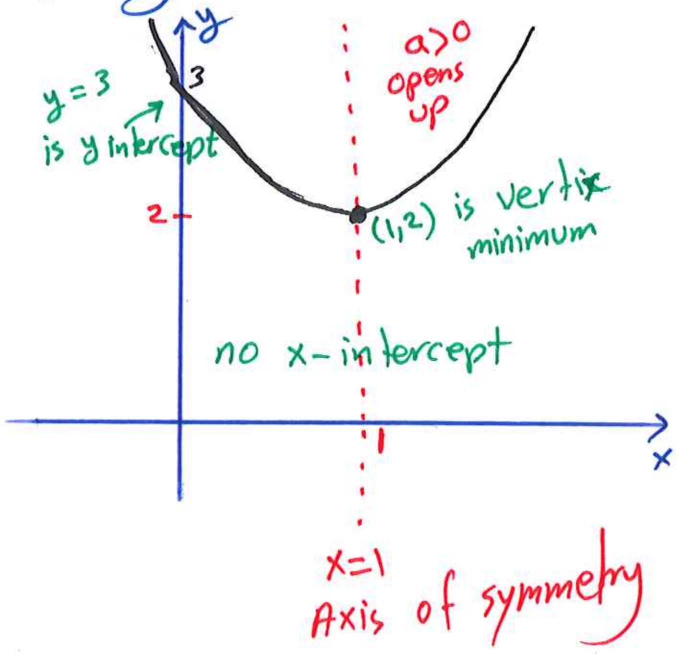


Remark If $D = b^2 - 4ac < 0$ then no x-intercept

3. $y = (x-1)^2 + 2$

same as $y = x^2$ but shift to right by 1 unit and shift up by 2 units





or $y = (x-1)^2 + 2$

$= x^2 - 2x + 1 + 2$

$= x^2 - 2x + 3 \Rightarrow a = 1 > 0 \Rightarrow$ parabola opens up

$\Rightarrow$ vertex is minimum

$b = -2, \; c = 3 \Rightarrow x = -\frac{b}{2a} = -\frac{-2}{2(1)} = 1$ is axis of symmetry

$\Rightarrow$ vertex is $(1, f(1)) = (1, 2)$

x-intercept $\Rightarrow y = 0 \Rightarrow (x-1)^2 + 2 = 0 \Rightarrow (x-1)^2 = -2$ not possible

$\Rightarrow$ no x-intercept

since $\Delta = b^2 - 4ac$

$= (-2)^2 - 4(1)(3)$

$= 4 - 12$

$= -8$

is negative

y-intercept $\Rightarrow x = 0 \Rightarrow y = 3$

(4) $f(x) = 4x - 2x^2 - 4$

$= -2x^2 + 4x - 4$

$a = -2 < 0 \Rightarrow$ parabola opens down

$\Rightarrow$ vertex is maximum

$b = 4$, $c = -4$ $\Rightarrow x = -\frac{b}{2a} = -\frac{4}{2(-2)} = \frac{-4}{-4} = 1$ is the axis of symmetry

$\Rightarrow$ vertex is $(x, y) = (1, f(1))$

$= (1, -2)$

x-intercept $\Rightarrow y = 0 \Rightarrow -2x^2 + 4x - 4 = 0$

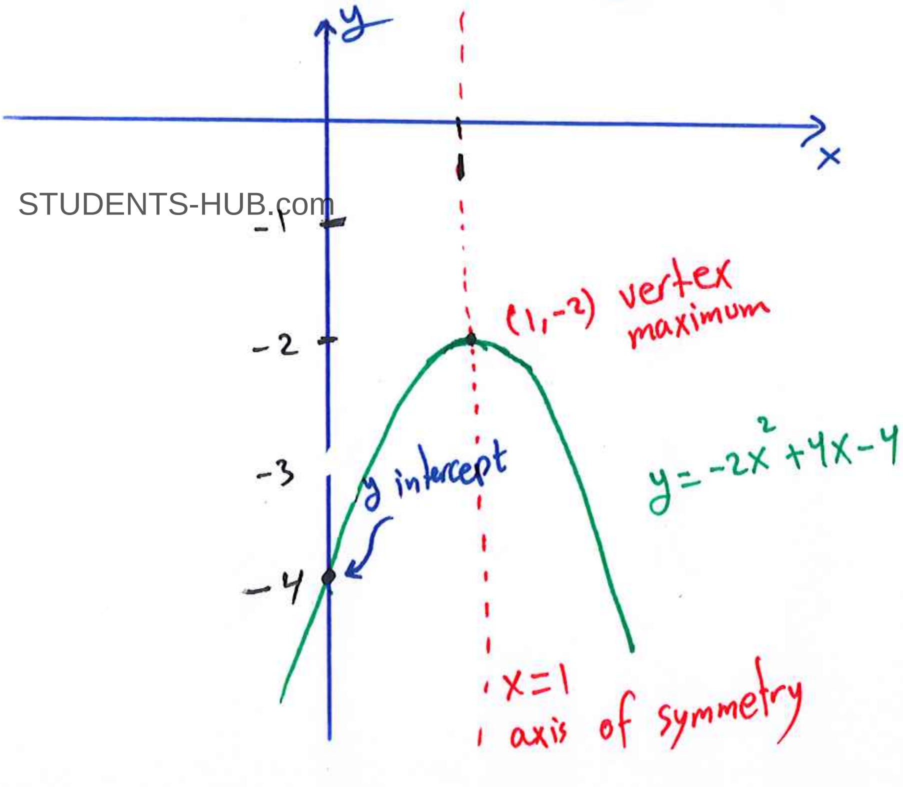
$D = b^2 - 4ac$

$= (4)^2 - 4(-2)(-4)$

$= 16 - 32$

$= -16 \Rightarrow$ no x-intercept

y-intercept $\Rightarrow x = 0 \Rightarrow y = -4$



69

(5) $x^2 + y = 8 + 2x$

$$y = -x^2 + 2x + 8$$

$a = -1 < 0 \Rightarrow$ parabola opens down $\Rightarrow$ vertex is maximum

$b = 2, c = 8 \Rightarrow x = -\frac{b}{2a} = -\frac{2}{2(-1)} = 1$ is axis of symmetry

$\Rightarrow$ vertex is $(1, f(1)) = (1, 9)$

x-intercept $\Rightarrow y = 0 \Rightarrow -x^2 + 2x + 8 = 0 \Rightarrow \Delta = b^2 - 4ac$

$$= (2)^2 - 4(-1)(8)$$
$$= 4 + 32$$
$$= 36 \text{ positive}$$

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-2 + \sqrt{36}}{2(-1)} = \frac{-2+6}{-2} = \frac{4}{-2} = -2$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-2 - \sqrt{36}}{2(-1)} = \frac{-2-6}{-2} = \frac{-8}{-2} = 4$$

y-intercept $\Rightarrow x = 0 \Rightarrow y = 8$