Birzeit University Faculty of Engineering Dep. of Electrical Engineering

Probability ENEE 331 Homework (3) Single Random Variables

P1) Let X and Y be two discrete Random Variable with a Joint-PdF given in the table below:

	Y = 0	Y = 1	f(x)
X = -1	0	0.25	0.25
X = 0	0.25	0.25	0.5
X = 1	0	0.25	0.25
f(y)	0.25	0.75	1

Are X and Y Correlated? Explain in details.

$$E[XY] = \sum_{0}^{1}\sum_{-1}^{1} XYf(x,y) = -1*0*0 + 0*0*0.25 + \cdots ... + 1*1*0.25 = 0$$

$$E[X] = \sum_{-1}^{1} xf(x) = -1*0.25 + 0*0.5 + 1*0.25 = 0$$

$$cov(X,Y) = E[XY] - \mu_x\mu_y = 0 - 0*0.75 = 0$$

$$\rho_{xy} = \frac{cov(X,Y)}{\sigma_x\sigma_y} = \frac{0}{\sigma_x\sigma_y} = 0$$

P2) Suppose continuous r.v.s $(X,Y)\in R^2$ have joint pdf

a) What is the cumulative distribution function of X?
b) Determine the marginal pdfs for X and Y?
c) Are X and Y independent?

$$f(x,y) = \begin{cases} \frac{1}{\pi} & for\_x^2 + y^2 \le 1 \\ 0 & otherwise \end{cases}$$

$x^2 + y^2 \le 1$ means $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and for x the same

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

Cumulative distribution function of X

$$F(x,y) = \int_{-\sqrt{1-y^2}}^{y}\int_{-\sqrt{1-x^2}}^{x} \frac{1}{\pi}dxdy = \int_{-\sqrt{1-y^2}}^{y} \frac{x}{\pi}\Big|_{-\sqrt{1-x^2}}^{x} dy =$$

$$\frac{x+\sqrt{1-x^2}}{\pi} y\Big|_{-\sqrt{1-y^2}}^{y} = \frac{x+\sqrt{1-x^2}}{\pi}(y + \sqrt{1-y^2})$$

Marginal pdfs for X and Y

$$f(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy = \frac{2}{\pi}\sqrt{1-x^2}$$

$$f(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx = \frac{2}{\pi}\sqrt{1-y^2}$$

$$f(x)f(y) \ne f(x,y)$$

$$\frac{2}{\pi}\sqrt{1-x^2} * \frac{2}{\pi}\sqrt{1-y^2} \ne \frac{1}{\pi}$$

So X and Y are not independent.

P3) Suppose the continuous random variables X and Y have the following joint probability density function:

$$f(x, y) = 3/2$$

for $x^2 \le y \le 1$ and $0 < x < 1$. What is the conditional distribution of Y given $X = x$?

$$f(y \mid x) = \frac{f(x, y)}{f(x)} = \frac{3}{2f(x)}$$

To find f(x) the integral must be with respect of y witch's limited by $x^2 \le y \le 1$

$$f(x) = \int_{x^2}^{1} \frac{3}{2} dy = \frac{3y}{2} \Big|_{x^2}^{1} = \frac{3(1 - x^2)}{2}$$

$$f(y \mid x) = \frac{3}{2\frac{3(1 - x^2)}{2}} = \frac{1}{(1 - x^2)}$$

P4) Prove that Var(X+Y) = Var(X) + 2 Cov(X,Y) + Var(Y)

Assume that the mean of X and Y is zero to make life easy, since the mean doesn't affect the variance:

By definition:

$$Var(x + y) = E[(x + y)^2] - (E[x + y])^2 = E[(x + y)^2] - 0$$

$$= E[x^2] + E[2xy] + E[y^2]$$

$$= E[x^2] + 2E[xy] + E[y^2]$$

$$= VAR(x) + 2Cov(x, y) + VAR(y) + E(x)^2 + E(y)^2$$

$$= VAR(x) + 2Cov(x, y) + VAR(y)$$

P5) The joint density of X and Y is given by

$$f_{xy}(x, y) = \begin{cases} e^{-(x+y)} & 0 < x, 0 < y \\ 0 & otherwise \end{cases}$$

Find P(X/Y≤a) if a positive constant.

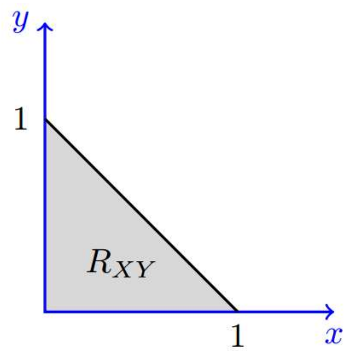
$$F(a) = P(\frac{X}{Y} \le a) = \int_0^\infty \int_{\frac{x}{a}}^\infty e^{-(x+y)} dy dx = \int_0^\infty e^{-x} \frac{e^{-y}}{-y} \Big|_{\frac{x}{a}}^\infty dx = \int_0^\infty e^{-x} \frac{e^{-\frac{x}{a}}}{\frac{x}{a}} dx = \frac{a}{a+1}$$

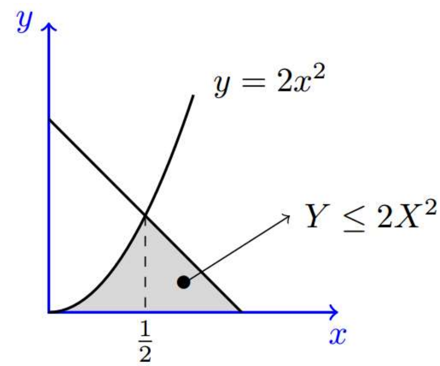
$$f(x) = \frac{dF(x)}{dx} = \frac{1}{(x+1)^2}$$

P6) The joint density of X and Y is given by

$$f_{xy}(x,y)=\begin{cases} cx+1 & 0<x, 0<y, x+y\le 1 \\ 0 & otherwise \end{cases}$$

a) Find the constant c.
b) Find $P(Y \le 2X^2)$
From $0<x, 0<y, x+y\le 1$ the left figure shows the region $y \le 1-x$
From $Y\le 2X^2$ the right figure shows the region $y \le 2x^2$





$$\int_0^\infty\int_0^\infty f(x,y)dydx = 1 = \int_0^1\int_0^{1-x}(cx+1)dydx = \frac{1}{2}+\frac{1}{6}c$$

$c=3$

$$P(Y<2X^2)=\int_0^{0.5}\int_0^{2X^2}(3x+1)dydx+\int_{0.5}^1\int_0^{(1-x)}(3x+1)dydx=$$

$$\int_0^{0.5}2x^2(3x+1)dy+\int_{0.5}^1(1-x)(3x+1)dydx$$

$=53/96$

Ahmad Alyan

2. Let X and Y be independent random variables, each exponentially distributed with mean $1/10$.

(a) Find $P(X \geq 5Y)$.

Solution. [This is a variation of the "light-bulb race" problem from class, which asked for the probability $P(X \geq Y)$, with X and Y having different exponential distributions.]
Since an exponential distribution with parameter λ has mean $\mu = 1/\lambda$, the parameter λ in the distribution of X and Y must be $\lambda = 1/(1/10) = 10$, and the joint c.d.f. is

$$f(x, y) = f_X(x) f_Y(y) = 10e^{-10x} 10e^{-10y}, \quad 0 < x < \infty, 0 < y < \infty.$$

Hence,

$$\begin{aligned} P(X \geq 5Y) &= \int_{y=0}^{\infty} \int_{x=5y}^{\infty} 10e^{-10x} 10e^{-10y} dy dx \\ &= \int_{y=0}^{\infty} e^{-5 \cdot 10y} 10e^{-10y} dy \\ &= \int_{y=0}^{\infty} 10e^{-60y} dy = \boxed{\frac{1}{6}} \end{aligned}$$

Remark: The parameter λ in the exponential distribution is **not** equal to the mean μ, but the reciprocal: Thus, $\lambda = 1/\mu = 1/(1/10) = 10$, and **not** $\lambda = 1/10$. While the final answer, $1/6$, ends up being the same with the latter choice of λ, the argument is not correct if one works with $\lambda = 1/10$.

(b) Let Z be the maximum (i.e., larger) of X and Y. Find the density, $f_Z(z)$, of Z.

Solution. [This is a variation on Problem 49(b), Chapter 6 from HW 9, and an illustration of the max/min trick that came up in class on several occasions.]
As usual, we first compute the c.d.f. $F_Z(z)$. We have

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(\max(X, Y) \leq z) = P(X \leq z, Y \leq z) \text{ (by the maximum trick)} \\ &= P(X \leq z) P(Y \leq z) \text{ (by the independence of } X \text{ and } Y \text{)} \\ &= (1 - e^{-10z})^2, \quad 0 \leq z < \infty. \end{aligned}$$

Now take the derivative to get the p.d.f.:

$$f_Z(z) = F_Z'(z) = 2(1 - e^{-10z}) 10e^{-10z} = \boxed{20\left(e^{-10z} - e^{-20z}\right), \quad 0 \leq z < \infty}$$

(c) Let $S = X + Y$. Find the density, $f_S(s)$, of S.

Solution. [This is just like HW Problem 27 in Chapter 6, but simpler, since the case distinction necessary in that hw problem is not needed here.]
As in the hw problem, we use the convolution formula for the density of a sum of two independent random variables: We have, for $s \geq 0$,

$$\begin{aligned} f_S(s) &= \int_{x=-\infty}^{\infty} f_X(x) f_Y(s - x) dx \\ &= \int_{x=0}^{s} 10e^{-10x} 10e^{-10(s-x)} dx \\ &= 100e^{-10s} \int_{x=0}^{s} dx \\ &= \boxed{100se^{-10s}, \quad 0 \leq s < \infty} \end{aligned}$$

3. Let X have uniform distribution on the interval $(0,1)$. Given $X = x$, let Y have uniform distribution on the interval $(0,x)$.

(a) Find the joint density of X and Y. Be sure to specify the range.

Solution. [This is a problem worked out in class.]

The given assumptions on X and Y are:

(1) X has uniform distribution on $[0,1]$, and

(2) given $X = x$, Y has uniform distribution on $(0,x)$.

This translates into

$$f_X(x) = 1, \quad 0 \le x \le 1, \tag{1}$$

$$f_{Y|X}(y|x) = \frac{1}{x}, \quad 0 \le y \le x. \tag{2}$$

Hence the joint density is

$$f(x,y) = f_{Y|X}(y|x) f_X(x) = \frac{1}{x} \cdot 1 = \boxed{\frac{1}{x}, \quad 0 \le x \le 1, 0 \le y \le x}$$

(b) Find the marginal density $f_Y(y)$ of Y. Be sure to specify the range.

Solution.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{x=y}^{1} \frac{1}{x} dx = \boxed{-\ln y, \quad 0 < y < 1},$$

Comments: As pointed out in class and in the solutions to several hw problems (e.g., Problem 8, Chapter 6, from HW 8), in computing marginal densities it is absolutely crucial to keep track of ranges of densities and to use these ranges in determining integration limits. In the above situation, integrating from $x = 0$ to $x = 1$ (instead of $x = y$ to $x = 1$) would be a fatal mistake, and would result in a nonsensical answer, namely, the integral $\int_0^1 \frac{1}{x} dx$, which does not exist.

(c) Find $E(XY)$.

Solution.

$$E(XY) = \iint xyf(x,y)dydx = \int_{x=0}^{1} \int_{y=0}^{x} xy\frac{1}{x} dydx$$
$$= \int_{x=0}^{1} \frac{x^2}{2} dx = \boxed{\frac{1}{6}}$$