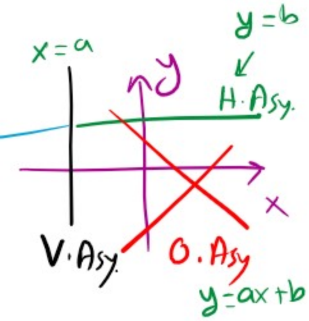
Q. How can we draw the rational function

$$f(x) = \frac{g(x)}{h(x)} \; ? \qquad g, h \text{ poly}$$

A. If we understand Asymptotes, then it helps to draw $f(x)$

$\Downarrow$ lines



$f(x) = \frac{g(x)}{h(x)}$ ✓ ✓

Asymptotes

- Horizontal Asy. (H. Asy) [1]
- Vertical Asy. (V. Asy) [2]
- Oblique Asy. (O. Asy) [3]

---

[1] H. Asy: $y = b$ is H. Asy for $f(x)$ if

(عدد ثابت) (عدد ثابت)

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Exp $f(x) = \frac{1}{x}$ Find H. Asy.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0 \Rightarrow \boxed{y = 0} \text{ H. Asy.}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = 0$$

(2) V. Asy : $x = a$ is V. Asy for $f(x)$ if

check zeros of $h(x)$

$$\lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

Or

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty$$

Exp $f(x) = \frac{1}{x}$ Find V. Asy.

$x = 0$

check $x = 0$

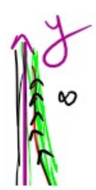
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{\text{small } +} = +\infty$$

so $x = 0$ is V. Asy.

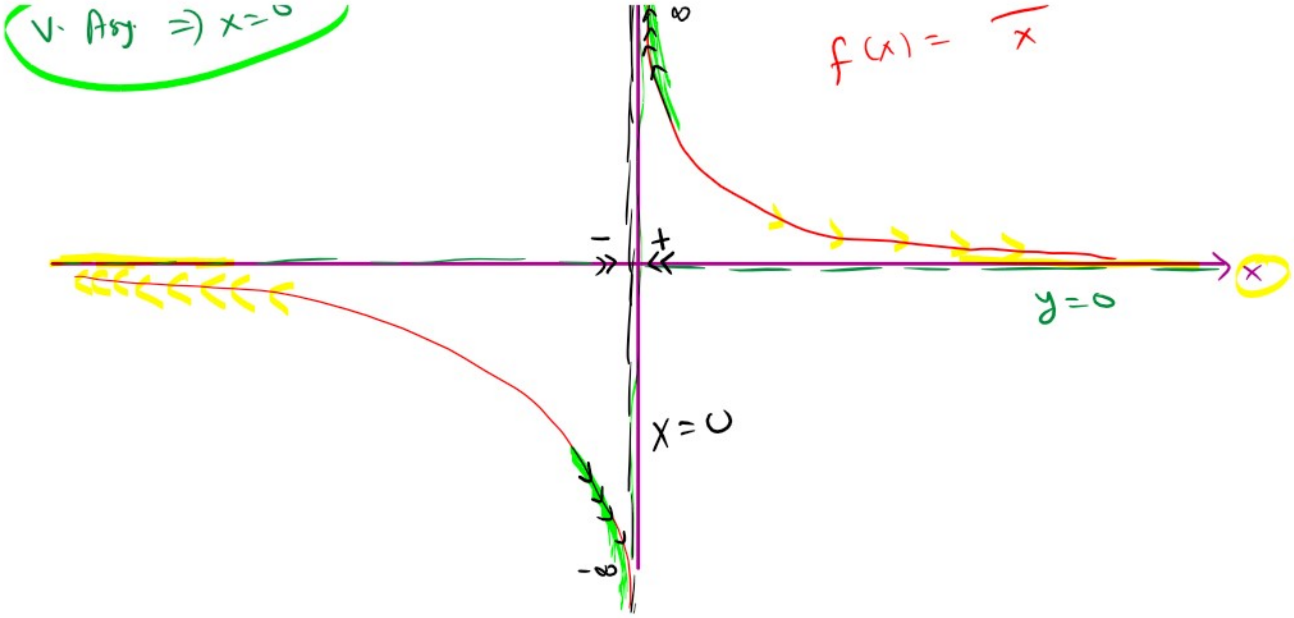
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = \frac{1}{\text{small } -} = -\infty$$

$\boxed{\text{H. Asy} \Rightarrow y = 0}$

V. Asy. $\Rightarrow x = 0$



$f(x) = \frac{1}{x}$



3) O. Asy: $y = ax + b$ is O. Asy. if

مائل

degree $g(x)$ is one greater than degree $h(x)$

$f(x) = \frac{g(x)}{h(x)} = \frac{x^{2}}{x+1}$ ✓

$g(x)$: numerator

$h(x)$: denomenator

✓ $\exists$ oblique Asy. $\Rightarrow \nexists$ H. Asy. $\rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x+1} = \infty$

$f(x) = \frac{1}{x}$;

$$\boxed{\nexists \text{ O. Asy} \Longrightarrow \exists \text{ H. Asy.}}$$

$f(x) = \frac{x^4}{x+1}$ has no O. Asy.

has no. H. Asy

To find O. Asy. $\Rightarrow$ we use long division
القسمة الطويلة

Exp ① $f(x) = \dfrac{x^3+1}{x^2-1}$ Find All Asy.
Sketch.

② $f(x) = \dfrac{x+1}{x-1}$

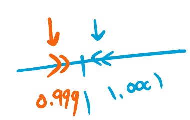
② H. Asy. $\Rightarrow$ $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x+1}{x-1} = \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$

$= \dfrac{1+0}{1-0} = \dfrac{1}{1} = 1$

$\boxed{y=1}$ H. Asy.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x+1}{x-1} = 1$

V. Asy $\Rightarrow$ check $x=1$



$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} \dfrac{x+1}{x-1} = \dfrac{2}{\text{small}+} = \infty$

$\boxed{x=1}$ is V. Asy

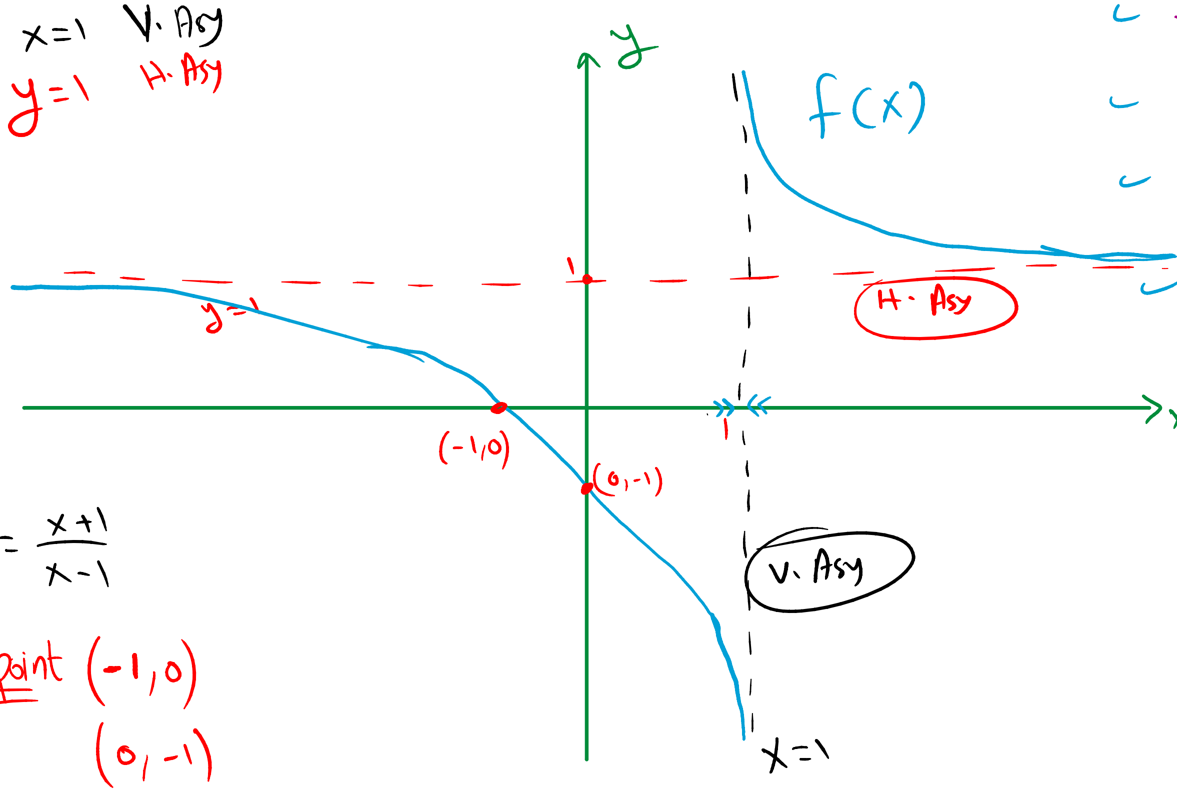
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+1}{x-1} = \frac{2}{\text{small } -} = -\infty$$

3) f has no O. Asy. since f has H. Asy.

$$f(x) = \frac{\overset{1}{x}+1}{\overset{1}{x}-1} \quad \times \quad \nexists \text{ O. Asy.}$$

$x=1$ V. Asy

$y=1$ H. Asy



$\lim_{x\to\infty} f(x) = 1$

$\lim_{x\to-\infty} f(x) = 1$

$\lim_{x\to 1^+} f(x) = \infty$

$\lim_{x\to 1^-} f(x) = -\infty$

$f(x) = \frac{x+1}{x-1}$

Keypoint $(-1,0)$

$(0,-1)$

H. Asy $\quad \lim_{x\to\infty} f(x) =$ عدد $\quad$ ✗

O. Asy. [illegible] $\Rightarrow \lim \left(\quad \right) = \infty$ or $-\infty$

O. Asy. $\dfrac{\text{درجة البسط أكبر بواحد}}{\text{من المقام}} \Rightarrow \lim\limits_{x \to \infty} (\quad) = \infty$ or $-\infty$

If f has H. Asy $\Rightarrow$ $\nexists$ O.A

درجة البسط

$$f(x) = \frac{x^{1}}{x^{1}+1}$$

درجة المقام

درجة البسط أقل

$$= \frac{\quad}{x^2+5}$$

درجة المقام

$\nexists$ O.A.

H. + V. ✓

V. + O. ✓

H. + O. ✗

---

(2) $f(x) = \dfrac{x^{3}+1}{x^{2}-1}$ $\quad\exists$ O. Asy $\Rightarrow$ $\nexists$ H. Asy.

(1) $\nexists$ H. Asy

O. Asy $\Rightarrow$ long division

2) $\exists$ O. Asy $\Rightarrow$ long division

$$\begin{array}{r} x \\ x^2-1 \overline{) x^3+1} \\ -x^3+x \\ \hline x+1 \end{array}$$

$$f(x) = \frac{x^3+1}{x^2-1} = x + \frac{x+1}{x^2-1}$$

O. As

$y = x$ is O. Asy

3) V. Asy Check $x^2-1=0$

$x^2 = 1$

$\boxed{x = \pm 1}$

1.001

1

Check $\boxed{x = 1}$ $\Rightarrow \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \frac{x^3+1}{x^2-1} = \frac{2}{\text{small}+} = \infty$

$\Rightarrow$ $x = 1$ is V. As

0.999

1

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^3+1}{x^2-1} = \frac{2}{\text{small}-} = -\infty$$

check $\boxed{x = -1}$ $\Rightarrow \lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} \frac{x^3+1}{x^2-1} = \lim\limits_{x \to -1^+} \frac{3x^2}{2x}$ $\frac{0}{0}$

$$= \frac{3}{-2} = -\frac{3}{2}$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x^3+1}{x^2-1} = \lim_{x \to -1^-} \frac{3x^2}{2x} = -\frac{3}{2}$$

Hence, $x = -1$ is not V. Asy

So $x = -1$ is removable discont.

$y = x$ O. Asy.

$x = 1$ V. Asy

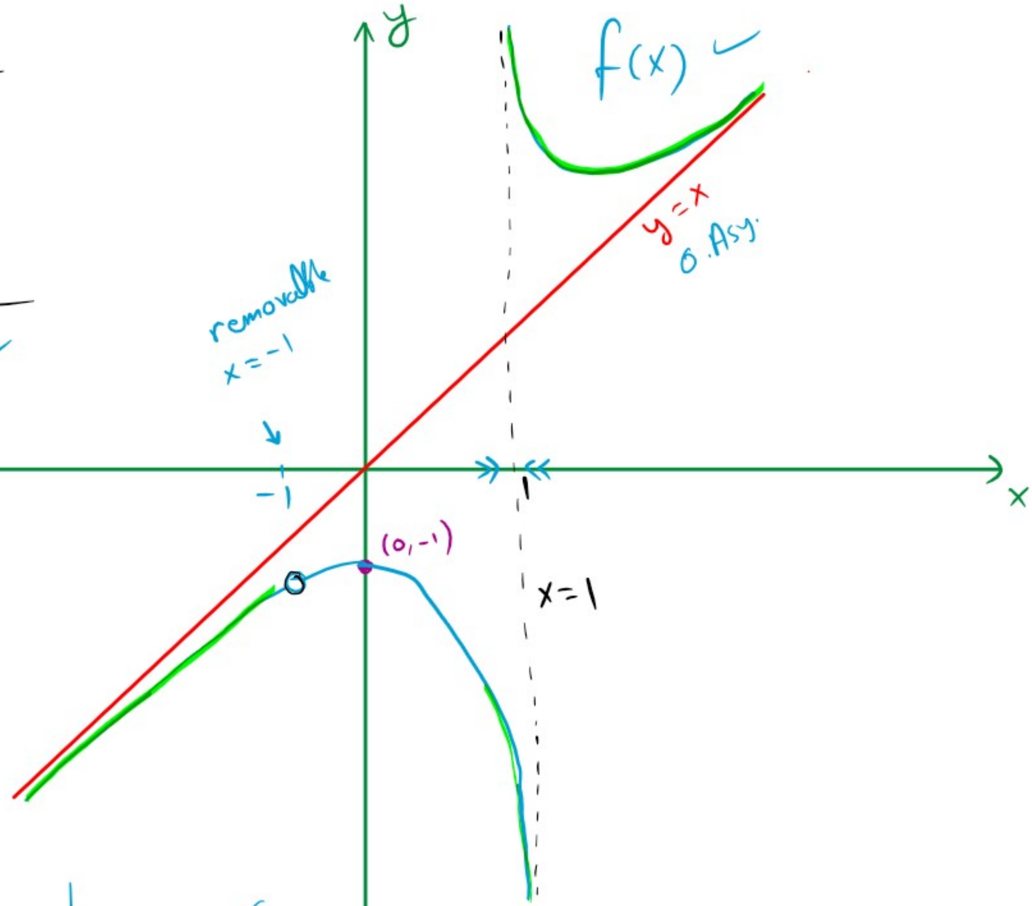
$$f(x) = \frac{x^3+1}{x^2-1}$$

$D(f) = \mathbb{R} \setminus \{\pm 1\}$

$\lim_{x \to 1^+} f(x) = \infty$

$\lim_{x \to 1^-} f(x) = -\infty$

Keypoint $(0,-1)$



Find cont. extension for f(x) at $x = -1$

$$F(x) = \begin{cases} f(x) & \text{if } x \neq -1 \\ \lim_{x \to -1} f(x) & \text{if } x = -1 \end{cases} = \begin{cases} \frac{x^3+1}{x^2-1} & \text{if } x \neq -1 \\ -\frac{3}{2} & \text{if } x = -1 \end{cases}$$

$x \to -1$

$\to$ cont. at $x=-1$

$$f(x) = \frac{2x^3+1}{x^2-5} \quad \text{Find } \underline{\text{O. Asy}}$$

$$\frac{2x^3}{x^2} = 2x$$

$$= 2x + \frac{10x+1}{x^2-5}$$

$$\begin{array}{r|l} & 2x \\ x^2-5 & 2x^3+1 \\ & -2x^3+10x \\ \hline & 10x+1 \end{array}$$

$y = 2x$ ✓

✓ $$\lim_{x \to \text{عدد}} \frac{\square}{\square} = \frac{\text{رقم غير الصفر}}{\text{صفر}} \begin{cases} \infty \to \dfrac{1}{\text{small}+} \\ -\infty \to \dfrac{1}{\text{small}-} \end{cases}$$

$$\frac{3}{0} \begin{cases} \infty \\ -\infty \end{cases}$$

Exp

$$\lim_{x\to 2} \frac{3}{x-2} = \text{DNE}$$

$$\lim_{x\to 2^+} \frac{3}{x-2}$$

2.001

2

$$\lim_{x\to 2^-} \frac{3}{x-2}$$

$$\frac{3}{\text{small}-}$$

1.99

2

$x \to 2^+$

$$\frac{3}{\text{small}+} = \infty$$

$$\frac{-}{\text{small}-} = -\infty$$

2

$$\lim_{x \to \text{رقم}} \frac{\square}{\square}$$

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

removable disc. $\Rightarrow$ not V. Asy.

Exp

$$\lim_{x \to -1^+} \frac{x}{x+1} = \cancel{\frac{-1}{0}} = \infty$$

-0.999

-1

$$\frac{1}{\text{small}+} = \infty$$

$$\frac{\text{رقم غير الصفر}}{\text{صفر}} \to \begin{cases} \text{small}+ \\ \text{small}- \end{cases}$$

$\frac{0}{0}, \frac{\infty}{\infty} \longrightarrow$ نطبق L'Hopital Rule

Remark: The graph of $f(x)$ might cross the Asy.

Exp $f(x) = \frac{\sin x}{x}$ Find Asy. sketch f

① H. Asy.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sin x}{x}$$

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\underbrace{-\lim_{x\to\pm\infty} \frac{1}{x}}_{0} \le \underbrace{\lim_{x\to\pm\infty} \frac{\sin x}{x}}_{0 \text{ by S.T}} \le \underbrace{\lim_{x\to\pm\infty} \frac{1}{x}}_{0}$$

$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0_{\&1} \Rightarrow \boxed{y=0 \text{ is H. Asy}}$$

$\nexists$ O. Asy.

② V. Asy. $f(x) = \frac{\sin x}{x}$ $\Rightarrow$ check $\boxed{x=0}$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\cos x}{1} = \cos 0 = 1 \qquad \frac{0}{0}$$

مش ∞, $-\infty$ X

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\cos x}{1} \qquad \frac{0}{0}$$

$$= \cos 0 = 1 \qquad X$$

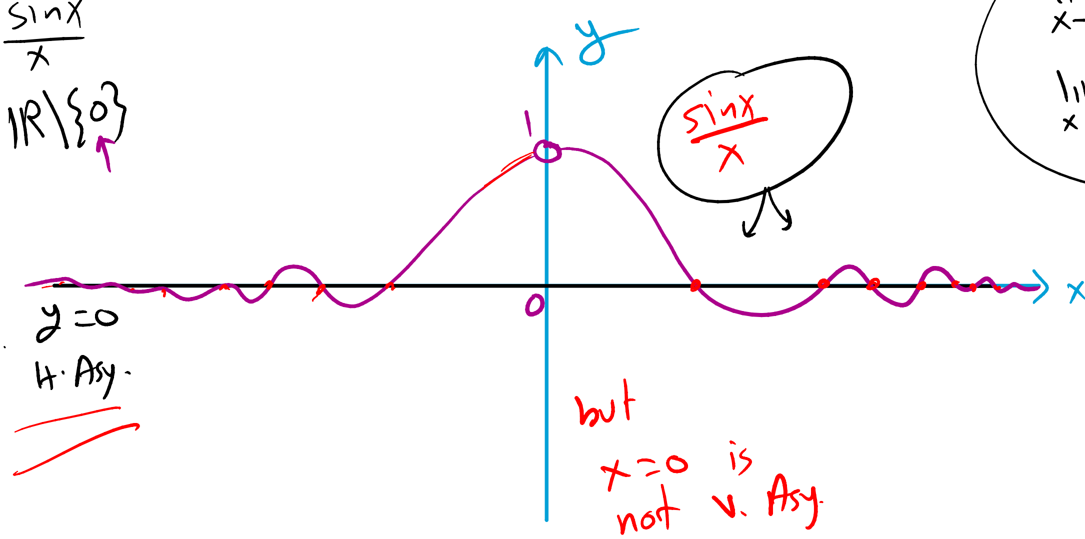
$\boxed{x = 0 \text{ is not V. Asy}}$

$\Downarrow$

$x = 0$ removable disc.

$f(x) = \frac{\sin x}{x}$

$D(f) = \mathbb{R} \setminus \{0\}$



$\lim_{x\to 0^+} f(x) = 1$

$\lim_{x\to 0^-} f(x) = 1$

$\Downarrow$

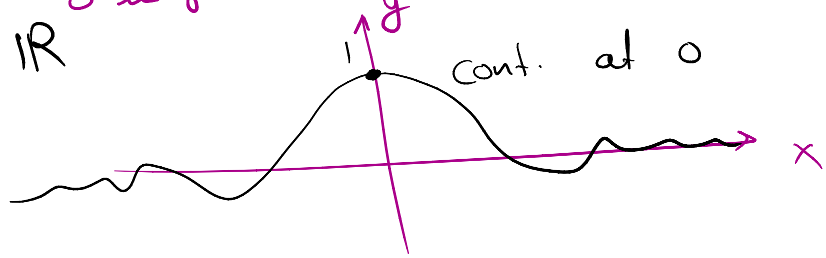
$\lim_{x\to 0} f(x) = 1$

$x = 0$ removal

Find cont. extension $\Rightarrow$ $F(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ \boxed{1} & \text{if } x = 0 \end{cases}$

if $x = 0$

$D(F) = \mathbb{R}$ 0 is fine



$$f(x) = \frac{x-2}{x^2+1} \Rightarrow \text{Keypoint } \underline{\underline{(2,0)}},\ (0,-2)$$

0, 1, 2, -2.

$g(x) =$.