7.7 Hyperbolic Functions

* $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$ * $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

Identities :

* $\cosh^2 x - \sinh^2 x = 1$
* $\sinh 2x = 2 \sinh x \cosh x$
* $\cosh 2x = \cosh^2 x + \sinh^2 x$
* $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
* $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
* $\tanh^2 x = 1 - \operatorname{sech}^2 x$
* $\coth^2 x = 1 + \operatorname{csch}^2 x$

Derivatives of hyperbolic functions:

1) $\dfrac{d}{dx}(\sinh u) = \cosh u \cdot \dfrac{du}{dx}$
2) $\dfrac{d}{dx}(\cosh u) = \sinh u \cdot \dfrac{du}{dx}$
3) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \cdot \dfrac{du}{dx}$
4) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \cdot \dfrac{du}{dx}$
5) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \cdot \tanh u \cdot \dfrac{du}{dx}$
6) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \cdot \coth u \cdot du$

Integral formulas for hyperbolic functions:

1) $\int \sinh u \, du = \cosh u + c$

2) $\int \cosh u \, du = \sinh u + c$

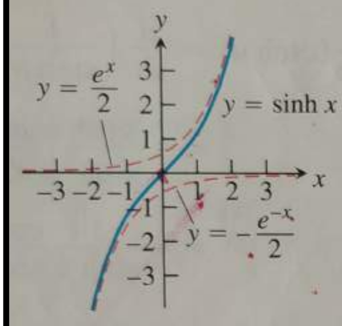
3) $\int \operatorname{sech}^2 u \, du = \tanh u + c$

4) $\int \operatorname{csch}^2 u \, du = -\coth u + c$

5) $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + c$

6) $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + c$

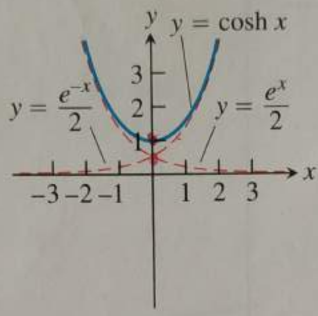
ABLE 7.5 The six basic hyperbolic functions



(a)

Hyperbolic sine:

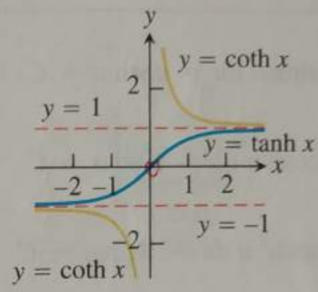
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



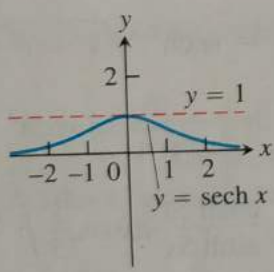
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

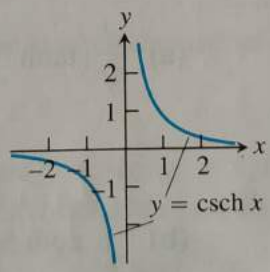
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

For any real number u, we know the point with coordinates $(\cos u, \sin u)$ lies on the unit circle $x^2 + y^2 = 1$. So the trigonometric functions are sometimes called the *circular* functions. Because of the first identity

$\boxed{2}$ Consider $\sinh x = \frac{4}{3}$. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1) $\cosh^2 x - \sinh^2 x = 1$

$\cosh^2(x) - \left(\frac{4}{3}\right)^2 = 1 \rightarrow \cosh^2 x = 1 + \left(\frac{4}{3}\right)^2 =$

$= 1 + \frac{16}{9} = \frac{25}{9}$

$\cosh x = \sqrt{\frac{25}{9}} = \frac{5}{3}$

2) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{4/3}{5/3} = \frac{4}{5}$

3) $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{5/3} = \frac{3}{5}$

4) $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{1}{4/3} = \frac{3}{4}$

5) $\coth x = \frac{1}{\tanh x} = \frac{1}{4/5} = \frac{5}{4}$ or $\frac{\cosh x}{\sinh x}$

$\boxed{6}$ Rewrite the expression in terms of exponentials

$\sinh(2\ln x)$

$\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2}$

$= \frac{x^2 - x^{-2}}{2} = \frac{x^2 - \frac{1}{x^2}}{2} = \frac{2x^4 - 1}{2x^2}$

$\boxed{10}$ Rewrite the expression in terms of exponentials

$$\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$$

$$\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$$

$$= \ln[(\cosh x + \sinh x)\cdot(\cosh x - \sinh x)]$$

$$= \ln[\cosh^2 x - \sinh^2 x] = \ln(1) = 0$$

Note that $\ln a + \ln b = \ln(a \cdot b)$

$\boxed{15}$ find the derivative of y with respect to t.

$$y = 2\sqrt{t}\ \tanh\sqrt{t}$$

$$y' = 2\sqrt{t}\cdot(\tanh\sqrt{t})' + \tanh\sqrt{t}\cdot(2\sqrt{t})'$$

$$y' = 2\sqrt{t}\cdot \operatorname{sech}^2\sqrt{t}\cdot\frac{1}{2\sqrt{t}} + \tanh\sqrt{t}\cdot 2\cdot\frac{1}{2\sqrt{t}}$$

$$y' = \operatorname{sech}^2\sqrt{t} + \frac{\tanh\sqrt{t}}{\sqrt{t}}$$

$\boxed{20}$ find the derivative of y. $y = \operatorname{csch}\theta(1 - \ln\operatorname{csch}\theta)$

$$y' = \operatorname{csch}\theta\cdot(1 - \ln\operatorname{csch}\theta)' + (1 - \ln\operatorname{csch}\theta)\cdot(\operatorname{csch}\theta)'$$

$$y' = \operatorname{csch}\theta\left(0 - \frac{(\operatorname{csch}\theta)'}{\operatorname{csch}\theta}\right) + (1 - \ln\operatorname{csch}\theta)\cdot -\operatorname{csch}\theta\coth\theta$$

$$y' = \operatorname{csch}\theta\left(- \frac{-\operatorname{csch}\theta\coth\theta}{\operatorname{csch}\theta}\right) \mp \operatorname{csch}\theta\coth\theta(1 - \ln\operatorname{csch}\theta)$$

$$y' = \cancel{\operatorname{csch}\theta\coth\theta} - \cancel{\operatorname{csch}\theta\coth\theta} + \operatorname{csch}\theta\coth\theta\ln\operatorname{csch}\theta$$

$$\boxed{y' = \operatorname{csch}\theta\coth\theta\cdot\ln\operatorname{csch}\theta}$$

$\boxed{22}$ $y = \ln \sinh v - \frac{1}{2} \coth^2 v$

find the derivative of y with respect to v.

$$y' = \frac{(\sinh v)'}{\sinh v} - \frac{1}{2}(2\coth v)(-\operatorname{csch}^2 v)$$

$$y' = \frac{\cosh v}{\sinh v} + \coth v \cdot \operatorname{csch}^2 v$$

$$y' = \coth v + \coth v \cdot \operatorname{csch}^2 v$$

$$= \coth v(1 + \operatorname{csch}^2 v) = \coth v(\coth^2 v)$$

$$\boxed{y' = \coth^3 v}$$

$\boxed{46}$ Evaluate $\int \coth\left(\frac{\theta}{\sqrt{3}}\right) d\theta$

$$= \int \frac{\cosh\left(\frac{\theta}{\sqrt{3}}\right)}{\sinh\left(\frac{\theta}{\sqrt{3}}\right)} d\theta$$

$\rightarrow$ let $V = \frac{\theta}{\sqrt{3}}$

$dV = \frac{d\theta}{\sqrt{3}}$

$$= \sqrt{3} \int \frac{\cosh v}{\sinh v} dv$$

$$= \sqrt{3} \ln|\sinh v| + c$$

$$= \sqrt{3} \ln\left|\sinh \frac{\theta}{\sqrt{3}}\right| + c$$

$$= \sqrt{3} \ln\left|\frac{e^{\frac{\theta}{\sqrt{3}}} - e^{-\frac{\theta}{\sqrt{3}}}}{2}\right| + c = \sqrt{3}\left[\ln\left(e^{\frac{\theta}{\sqrt{3}}} - e^{-\frac{\theta}{\sqrt{3}}}\right) - \ln 2\right] + c$$

$$= \sqrt{3}\left(\ln e^{\frac{\theta}{\sqrt{3}}} - e^{-\frac{\theta}{\sqrt{3}}}\right) - \sqrt{3} \ln 2 + c$$

✓ $\boxed{52}$ Evaluate $\displaystyle\int_0^{\ln 2} \tanh 2x \, dx$.

$$= \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x}\, dx = \frac{1}{2}\int_0^{\ln 2} \frac{2\sinh 2x}{\cosh 2x}\, dx$$

$$= \frac{1}{2} \ln|\cosh 2x| \Big|_0^{\ln 2}$$

$$= \frac{1}{2}\left[\ln(\cosh 2\ln 2) - \ln(\cosh 2(0)) \right]$$

$$= \frac{1}{2}\left[\ln(\cosh \ln 4) - \ln(\cosh 0) \right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{e^{\ln 4} + e^{-\ln 4}}{2}\right) - \ln\left(\frac{e^{0} + e^{-0}}{2}\right) \right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{4 + e^{\ln 4^{-1}}}{2}\right) + \ln\left(\frac{1+1}{2}\right) \right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{4 + 4^{-1}}{2}\right) + \ln(1) \right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{4 + \frac{1}{4}}{2}\right) + 0 \right]$$

$$= \frac{1}{2}\left[\ln \frac{17}{8} \right]$$

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$\boxed{54}$ Evaluate $\int_0^{\ln 2} 4 e^{-\theta} \sinh\theta \, d\theta$

$= \int_0^{\ln 2} 4 e^{-\theta} \left(\frac{e^{\theta} - e^{-\theta}}{2} \right) d\theta$

$= \int_0^{\ln 2} 4 \left(\frac{e^{0} - e^{-2\theta}}{2} \right) d\theta$

$= 2 \int_0^{\ln 2} \left(1 - e^{-2\theta} \right) d\theta$

$= 2 \left[\theta - \int e^{-2\theta} \, d\theta \right] \Big|_0^{\ln 2}$

$= 2 \left[\theta - \left(-\frac{1}{2}\right) \int -2 e^{-2\theta} \, d\theta \right] \Big|_0^{\ln 2}$

$= 2 \left[\theta + \frac{1}{2} e^{-2\theta} \right] \Big|_0^{\ln 2}$

$= 2 \left[\ln 2 + \frac{1}{2} e^{-2\ln 2} - \left(0 + \frac{1}{2} e^{-2(0)} \right) \right]$

$= 2 \left[\ln 2 + \frac{1}{2} e^{\ln 2^{-2}} - 0 - \frac{1}{2}(1) \right]$

$= 2 \left[\ln 2 + \frac{1}{2} (2^{-2}) - \frac{1}{2} \right]$

$= 2 \left[\ln 2 + \frac{1}{2} \left(\frac{1}{4} \right) - \frac{1}{2} \right]$

$= 2 \left[\ln 2 - \frac{3}{8} \right] = \boxed{\ln 4 - \frac{3}{4}}$

$\boxed{57}$ Evaluate $\int_1^2 \frac{\cosh(\ln t)}{t}\, dt$

Let $u = \ln t \rightarrow du = \frac{dt}{t}$

$$\int \frac{\cosh(\ln t)}{t}\, dt = \int \cosh(u)\, du$$

$$= \sinh u = \sinh(\ln t)$$

$$\int_1^2 \frac{\cosh(\ln t)}{t}\, dt = \sinh(\ln t) \Big|_1^2$$

$$= \sinh(\ln 2) - \sinh(\ln 1)$$

$$= \frac{e^{\ln 2} - e^{-\ln 2}}{2} - \left(\frac{e^{\ln 1} - e^{-\ln 1}}{2}\right)$$

$$= \frac{2 - e^{\ln 2^{-1}}}{2} - \left(\frac{1 - e^{\ln 1^{-1}}}{2}\right)$$

$$= \frac{2 - 2^{-1}}{2} - \left(\frac{1 - 1}{2}\right)$$

$$= \frac{2 - \frac{1}{2}}{2} - 0$$

$$= \frac{4 - 1}{4} = \boxed{\frac{3}{4}}$$

$\boxed{60}$ Evaluate $\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx$

$$= \int_0^{\ln 10} 4^{2}\left(\frac{\cosh x - 1}{\cancel{2}}\right) dx$$

$$= 2\int_0^{\ln 10} \cosh x - 1 \; dx$$

$$= 2\left[\sinh x - x\right]\Big|_0^{\ln 10}$$

$$= 2\left[\sinh \ln 10 - \ln 10 - (\sinh 0 - 0)\right]$$

$$= 2\left[\frac{e^{\ln 10} - e^{-\ln 10}}{2} - \ln 10 - \frac{e^{0} - e^{-0}}{2}\right]$$

$$= 2\left[\frac{e^{\ln 10} - e^{\ln 10^{-1}}}{2} - \ln 10\right]$$

$$= e^{\ln 10} - e^{\ln 10^{-1}} - \ln 10 = 10 - 10^{-1} - 2\ln 10$$

$$10 - \frac{1}{10} = \frac{100-1}{10} = \frac{99}{10} - 2\ln 10$$

$$= \boxed{\frac{99}{10} - \ln 100}$$