7.3 Exponential Functions.

* Inverse Equations for e^x and $\ln x$

  - $e^{\ln x} = x$ for $x > 0$
  - $\ln e^x = x$ for all x.

* If u is any differentiable function of x, then $\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$

* $\int e^u \, du = e^u + c$

* The natural exponential e^x obeys the followi[ng] laws:

  - $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
  - $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
  - $(e^{x_1})^r = e^{r x_1}$, if r is rational

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* The exponential function with base a is

$$a^x = e^{x \ln a}$$

* The number e as a limit:

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

* $\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$

* $\int a^u \, du = \frac{a^u}{\ln a} + c$

* for $a > 0$, $a \neq 1$, $\log_a x$ is the inverse of a^x

* $a^{\log_a x} = x$ and $\log_a a^x = x$

* $\log_a x = \frac{\ln x}{\ln a}$

* $\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$

Question 4. Solve for t $e^{x^2} e^{2x+1} = e^t$

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$$e^{x^2+2x+1} = e^t$$

$$x^2 + 2x + 1 = t$$

Question 14. Find the derivative of y with respect to x. $y = \ln(3\theta e^{-\theta})$

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$$y = \ln(3\theta e^{-\theta}) = \ln 3 + \ln\theta + \ln e^{-\theta}$$

$$y = \ln 3 + \ln\theta - \theta\ln e$$

$$y = \ln 3 + \ln\theta - \theta(1)$$

$$y' = 0 + \frac{1}{\theta} - 1 = \frac{1}{\theta} - 1$$

$$\boxed{y' = \frac{1}{\theta} - 1}$$

Question 21
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Find the derivative of y with respect to x $\quad y = e^{(\cos t + \ln t)}$

$$y = e^{\cos t} \cdot e^{\ln t} = e^{\cos t} \cdot t$$

$$y' = e^{\cos t} \cdot 1 + t \cdot e^{\cos t} \cdot -\sin t$$

$$y' = e^{\cos t} - t\, e^{\cos t} \sin t$$

Question 23.
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Find the derivative of y with respect to x. $\quad y = \int_0^{\ln x} \sin e^t \, dt$

$$y' = \sin e^{\ln x} \cdot (\ln x)' - \sin e^0 \cdot (0)'$$

$$y' = \sin(x) \cdot \frac{1}{x} - \sin(1) \cdot (0)$$

$$y' = \frac{\sin(x)}{x}$$

Question 26. Find $\frac{dy}{dx}$ $\ln xy = e^{x+y}$

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$$\rightarrow \ln x + \ln y = e^{x+y}$$

$$\frac{1}{x} + \frac{y'}{y} = e^{x+y}(1+y')$$

$$\frac{1}{x} + \frac{y'}{y} = e^{x+y} + y' e^{x+y}$$

$$\frac{y'}{y} - y' e^{x+y} = e^{x+y} - \frac{1}{x}$$

$$y'\left[\frac{1}{y} - e^{x+y}\right] = e^{x+y} - \frac{1}{x}$$

$$y'\left[\frac{1 - y e^{x+y}}{y}\right] = \frac{x e^{x+y} - 1}{x}$$

$$y' = \frac{\frac{x e^{x+y} - 1}{x}}{\frac{1 - y e^{x+y}}{y}}$$

$$y' = \frac{x e^{x+y} - 1}{x} \cdot \frac{y}{1 - y e^{x+y}}$$

$$y' = \frac{y(x e^{x+y} - 1)}{x(1 - y e^{x+y})}$$

Question 38 Evaluate $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} \, dr$

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$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} \, dr$$

Let $u = -\sqrt{r}$

$$\frac{du}{dr} = -\frac{1}{2\sqrt{r}}$$

$$-2\,du = \frac{dr}{\sqrt{r}}$$

$$= \int e^{u} \cdot -2\,du$$

$$= -2 \int e^{u} \, du$$

$$= -2\, e^{u} + c \quad = -2\, e^{-\sqrt{r}} + c$$

Question 41 Evaluate $\int \frac{e^{1/x}}{x^2} \, dx$

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$$\int \frac{e^{1/x}}{x^2} \, dx$$

Let $u = \frac{1}{x}$

$$\frac{du}{dx} = \frac{-1}{x^2}$$

$$-du = +\frac{dx}{x^2}$$

$$= \int e^{u} \cdot -du$$

$$= -\int e^{u} \, du$$

$$= -e^{u} + c$$

$$= -e^{1/x} + c$$

Question 48.

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Evaluate $\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos(e^{x^2})\, dx$

Let $u = e^{x^2}$

$\frac{du}{dx} = e^{x^2} \cdot 2x$

$du = 2x\, e^{x^2}\, dx$

$\int 2x\, e^{x^2} \cos(e^{x^2})\, dx$

$= \int \cos(u)\, du$

$= \sin u = \sin(e^{x^2})$

$\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos(e^{x^2})\, dx = \sin(e^{x^2}) \Big|_0^{\sqrt{\ln \pi}}$

$= \sin\left(e^{(\sqrt{\ln \pi})^2}\right) - \sin\left(e^{0^2}\right)$

$= \sin\left(e^{\ln \pi}\right) - \sin(1) = \sin(\pi) - \sin(1)$

Question 65. Find the derivative of y with respect to

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$y = 2^{\sin 3t}$

$y' = 2^{\sin 3t} \cdot \ln 2 \cdot (\sin 3t)'$

$= 2^{\sin 3t} \cdot \ln 2 \cdot (\cos 3t) \cdot (3)$

$y' = (3 \ln 2)(\cos 3t)(2^{\sin 3t})$

$\boxed{50}$ Evaluate the following integral.

$$\int \frac{dx}{1+e^{x}} \cdot \frac{e^{-x}}{e^{-x}} = \int \frac{e^{-x}}{e^{-x}+1}\, dx$$

Let $u = e^{-x}+1 \rightarrow du = -e^{-x}\, dx$

$$\int \frac{e^{-x}}{e^{-x}+1}\, dx = \int \frac{-du}{u} = \cancel{\int u^{-1}}\, du$$

$$= -\ln|u| + c = -\ln|e^{-x}+1| + c$$

$$= -\ln(e^{-x}+1) + c$$

$\boxed{52}$ Solve the initial value problem.

$$\frac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t}) \quad , \quad y(\ln 4) = \frac{2}{\pi}$$

$$\int \frac{dy}{dt}\, dt = \int e^{-t}\sec^2(\pi e^{-t})\, dt$$

$$y = \int e^{-t}\sec^2(\pi e^{-t})\, dt$$

let $u = \pi e^{-t} \rightarrow du = -\pi e^{-t} dt$

$$y = \int e^{-t} \sec^2(\pi e^{-t}) dt = \int \sec^2(u) \frac{du}{-\pi}$$

$$= \frac{-1}{\pi} \int \sec^2(u) du = -\frac{1}{\pi} \tan(u) + c$$

$$= \frac{-1}{\pi} \tan(\pi e^{-t}) + c$$

$y(\ln 4) = \frac{2}{\pi} \rightarrow$

$$\frac{2}{\pi} = -\frac{1}{\pi} \tan(\pi e^{-\ln 4}) + c$$

$$\frac{2}{\pi} = -\frac{1}{\pi} \tan(\pi e^{\ln 4^{-1}}) + c$$

$$\frac{2}{\pi} = \frac{-1}{\pi} \tan\left(\frac{\pi}{4}\right) + c$$

$$\frac{2}{\pi} = -\frac{1}{\pi} (1) + c$$

$$c = \frac{2}{\pi} + \frac{1}{\pi} = \frac{3}{\pi}$$

$$y = \frac{-1}{\pi} \tan(\pi e^{-t}) + \frac{3}{\pi}$$

Question 73. page 386

Find the derivative of y with respect to x. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right)$

remember that $\log_a b = \frac{\ln b}{\ln a}$

$$y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3}$$

$$y = \frac{\ln 3 \ \ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right)$$

$$y = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$$

$$y' = \frac{1}{x+1} - \frac{1}{x-1}$$

$$y' = \frac{x-1}{(x+1)(x-1)} - \frac{(x+1)}{(x-1)(x+1)}$$

$$y' = \frac{x-1-x-1}{(x-1)(x+1)} = \frac{-2}{(x-1)(x+1)}$$

Question 82. Find the derivative of y with respect

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to t. $y = t \log_3\left(e^{(\sin t)(\ln 3)}\right)$

$$y = t \log_3\left(e^{(\sin t)(\ln 3)}\right) = t \cdot \frac{\ln e^{(\sin t)(\ln 3)}}{\ln 3}$$

$$y = \frac{t\,(\sin t)(\ln 3)}{\ln 3} = t \sin t$$

$$y' = t \cdot \cos t + \sin t \cdot 1 = t \cos t + \sin t.$$

✓ Question 91. Evaluate $\int_2^4 x^{2x}(1+\ln x)\,dx$

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let $u = x^{2x}$

$\ln u = \ln x^{2x}$

$\ln u = 2x \ln x$

$\frac{du/dx}{u} = 2x \cdot \frac{1}{x} + \ln x \cdot 2$

$\frac{\frac{du}{dx}}{u} = 2 + 2\ln x$

$\frac{du}{dx} = (2+2\ln x)\,u$

$\frac{du}{dx} = (2+2\ln x)\,x^{2x}$

$$\int x^{2x}(1+\ln x)\,dx$$

$$= \frac{1}{2}\int 2x^{2x}(1+\ln x)\,dx$$

$$= \frac{1}{2}\int x^{2x}(2+2\ln x)\,dx$$

$$= \frac{1}{2}\int du$$

$$= \frac{1}{2}u = \frac{1}{2}x^{2x}$$

$$\int_2^4 x^{2x}(1+\ln x)\,dx = \frac{1}{2}x^{2x}\Big|_2^4$$

$$= \frac{1}{2}\left[4^{2(4)} - 2^{2(2)}\right] = \frac{1}{2}\left[4^8 - 2^4\right] = \boxed{32\ 760}$$

Question 92 Evaluate $\int \frac{x\, 2^{x^2}}{1+2^{x^2}}\,dx$

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$$\int \frac{x\, 2^{x^2}}{1+2^{x^2}}\,dx$$

Let $u = 2^{x^2}$

$\frac{du}{dx} = 2^{x^2} \cdot \ln 2 \cdot 2x$

$\frac{du}{2\ln 2} = x\, 2^{x^2}\,dx$

$$= \int \frac{1}{1+u} \cdot \frac{du}{2\ln 2}$$

$$= \frac{1}{2\ln 2} \int \frac{du}{1+u}$$

$$= \frac{1}{2\ln 2} \ln|1+u| + C$$

$$= \frac{1}{2\ln 2} \ln|1+2^{x^2}| + C$$

Question 98. Evaluate $\int_1^4 \frac{\log_2 x}{x}\,dx$

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$$\int \frac{\log_2 x}{x}\,dx = \int \frac{\ln x}{\ln 2 \cdot x}\,dx$$

$$= \frac{1}{\ln 2} \int \frac{\ln x}{x}\,dx$$

let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$$= \frac{1}{\ln 2} \int u\,du$$

$$= \frac{1}{\ln 2}\left(\frac{u^2}{2}\right) = \frac{1}{\ln 2}\left(\frac{(\ln x)^2}{2}\right)$$

$$\int_1^4 \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\,\frac{(\ln x)^2}{2}\Big|_1^4$$

$$= \frac{1}{2\ln 2}\left[(\ln 4)^2 - (\ln 1)^2\right]$$

$$= \frac{1}{2\ln 2}\left[(\ln 4)^2 - 0\right] = \frac{(\ln 4)^2}{2\ln 2}$$

$$= \frac{(\ln 4)^2}{\ln 4} = \boxed{\ln 4}$$

Question 106. Evaluate $\int \frac{dx}{x(\log_8 x)^2}$

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$$\int \frac{dx}{x(\log_8 x)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} = \int \frac{(\ln 8)^2}{x(\ln x)^2}\,dx$$

$$= (\ln 8)^2 \int \frac{du}{(u)^2}$$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$$= (\ln 8)^2 \int u^{-2}\,du$$

$$= (\ln 8)^2\,\frac{u^{-1}}{-1} + c = -(\ln 8)^2 \cdot \frac{1}{u} + c$$

$$= -(\ln 8)^2 \cdot \frac{1}{\ln x} + c$$

Question 108. Evaluate $\int_1^{e^x} \frac{1}{t}\,dt$

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$$\int_1^{e^x} \frac{1}{t}\,dt = \ln|t| \Big|_1^{e^x}$$

$$= \ln|e^x| - \ln|1| = \ln e^x - \ln 1$$

$$= x - 0 = x$$

Question 118. Use logarithmic differentiation to find $\frac{dy}{dx}$. $y = (\ln x)^{\ln x}$

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take $\ln y = \ln\left((\ln x)^{\ln x}\right)$

$$\ln y = (\ln x)(\ln(\ln x))$$

$$\frac{y'}{y} = (\ln x) \cdot (\ln(\ln x))' + \ln(\ln x) \cdot \frac{1}{x}$$

$$\frac{y'}{y} = \ln x \cdot \frac{(\ln x)'}{\ln x} + \ln(\ln x) \cdot \frac{1}{x}$$

$$\frac{y'}{y} = \frac{1}{x} + \frac{\ln(\ln x)}{x}$$

$$y' = y\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]$$

$$y' = (\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]$$