1 Regulator:-

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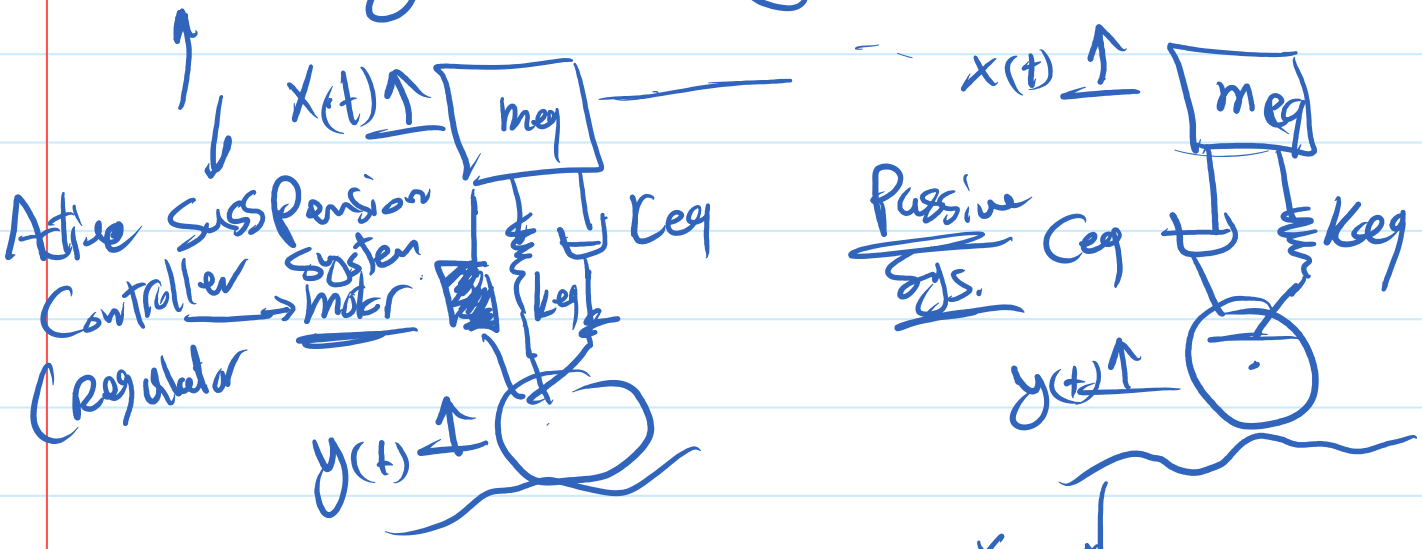
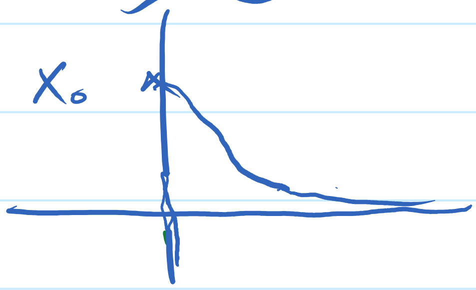
Regulator:- Controller aims to force the state to go to Zero or Eqp if the Sys is excited by IC or by external disturbance



$$\dot{x} = Ax + Bu \quad (1)$$

$$u = -Kx \quad \text{"Control action"} \quad (2)$$



K: gain matrix / vector , $K \in R^{n \times r}$

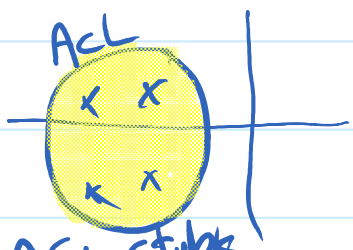
Sub (2) in (1)

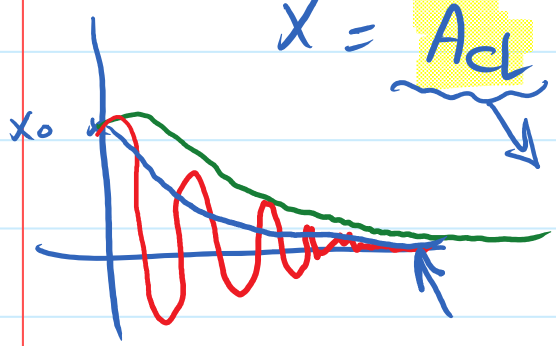
$$\dot{x} = Ax - BKx = (A - BK)\,x$$

Closed loop System matrix dynamic

$$\dot{x} = A_{cL}\,x$$

eig (A_{cL})





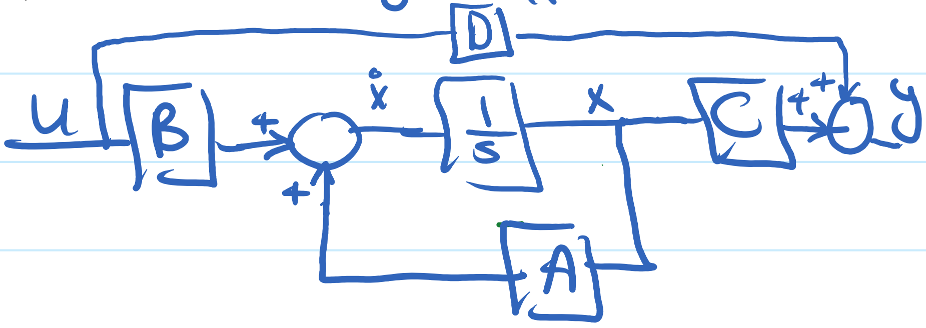
— overdamped $\xi > 1$

— critical damped $\xi = 1$

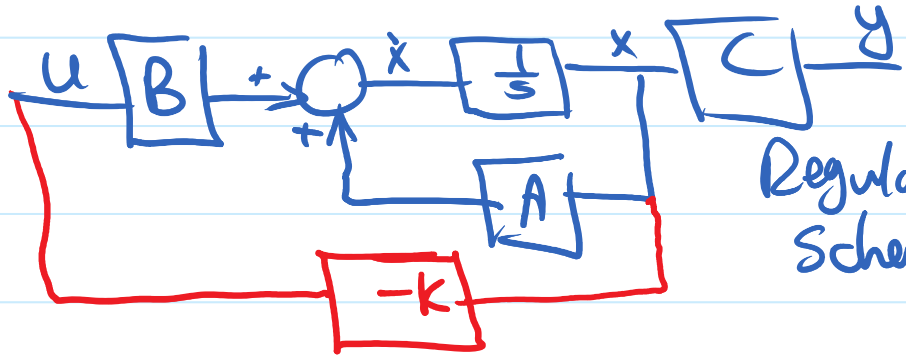
— underdamped $\xi < 1$

$\dot{x} = Ax + Bu$ —— (1)

$y = Cx + Du$



- if D is zero matrix, vector, scalar, the regulator scheme will become



Regulator Scheme

$u = -Kx$

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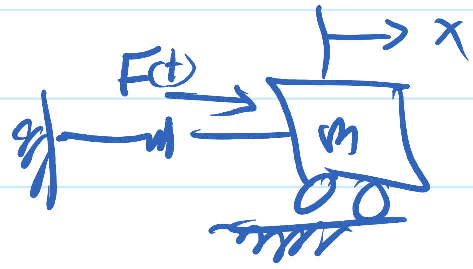
To find K :- (gain matrix) two approaches can be used

- Pole Placment
- LQR (Linear Quadratic Regulator)

Pole Placement :-

① The Sys. is Completely state Controllable
② The state vector (x) is measured or it is estimated

state vector $Z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$



Pole Placement Control Design

objective: The closed loop poles lie at $\mu_1, \ldots, \mu_n$ which are their desired locations

Method (1) :- (low order sys, $n \leq 3$)

(a) Check Controllability (✓)

(b) Define : $k = [k_1 \; \ldots \; k_n]$

(c) Substitue this gain in the desired char. polynomial equation $u = -kx$

$$|SI - A + Bk| = (s - \mu_1)(s - \mu_2) \ldots (s - \mu_n)$$

Closed loop char. eq. — desired char. eq.

$$s^3 + (d)s^2 + (e)s + (f) = s^3 + (a)s^2 + (b)s + c$$

* solve for $k_1, \ldots, k_n$ by equating the like power on both sides

$\dot{x} = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, Design Regulator if the required Performance $T_s = 1$ Sec, $\xi = 0.8$

$n = 2$

(1)

$M = [B \;\; AB]$

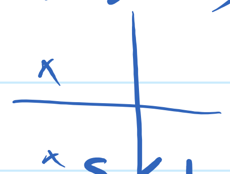
$= \left[\begin{matrix} 0 \\ 1 \end{matrix} \;\vdots\; \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 0 & 2 \\ 1 & 10 \end{bmatrix}$

$|M| = -2 \neq$ → Fully state Controllable

(2) $K = [k_1 \;\; k_2]$

(3) Desired eigen values ($T_s = 1$ Sec, $\xi = 0.8$)

$T_s = \dfrac{4}{\xi \omega_n} \Rightarrow \omega_n = \dfrac{4}{(1)(0.8)} = 5$ rad/s



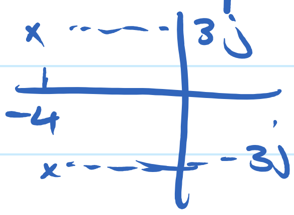
I. $S_{1,2} = -\xi\omega_n \pm \omega_n \sqrt{1-\xi^2}\, j$ ← $\xi < 1$ vibration underdamped

II. $S_{1,2} = -\omega_n$

III. $S_{1,2} = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$ ← overdamped $\xi > 1$

$S_{1,2} = -\xi\omega_n \pm \omega_n \sqrt{1-\xi^2} = -5 * 0.8 \pm 5\sqrt{1-(0.8)^2}\, j$

$= -4 \pm 3j$



(4) $|SI - A + BK|$

$\left| \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \;\; k_2] \right|$

2x1   1x2

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$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right|$$

2x1   1x2

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s-1 & -2 \\ -5+k_1 & s-10+k_2 \end{bmatrix} \right|$$

$$= (s-1)(s-10+k_2) - (-2)(-5+k_1) =$$

$$s^2 + (k_2 - 11)s + (2k_1 - k_2) \leftarrow$$ closed loop char. Eq

$\Rightarrow$ To find desired char. Eq. $M_{1,2} = -4 \pm 3j$

$$(s - M_1)(s - M_2) = (s+4-3j)(s+4+3j)$$

desired char. Eq. $\Rightarrow$ $= s^2 + 8s + 25$

(5) $k_2 - 11 = 8 \Rightarrow k_2 = 8 + 11 = 19$

$2k_1 - k_2 = 25 \Rightarrow 2k_1 - 19 = 25 \Rightarrow k_1 = 22$

Control Scheme

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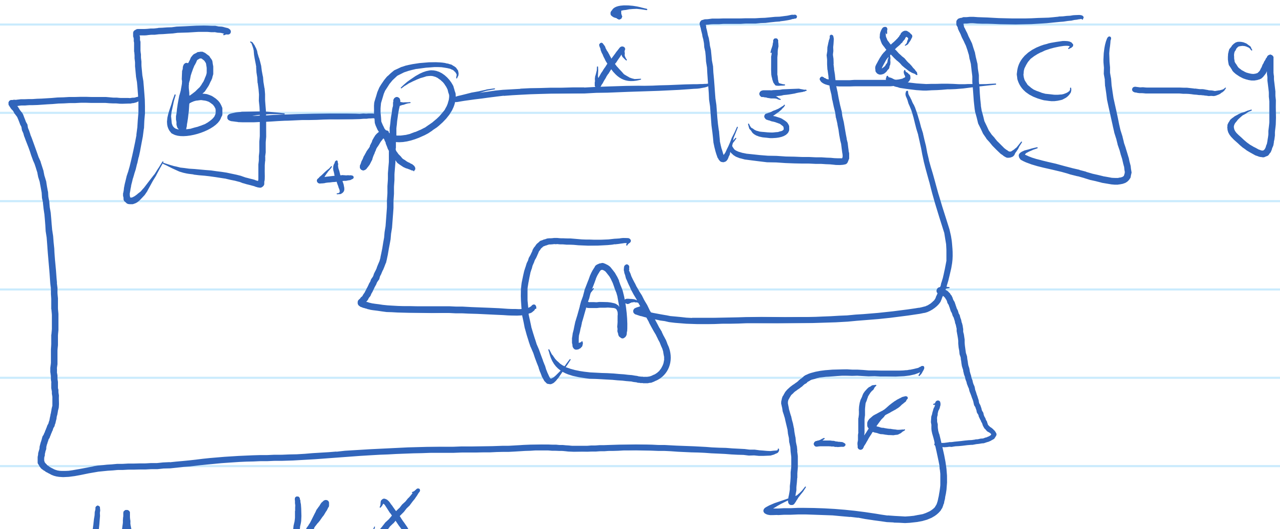
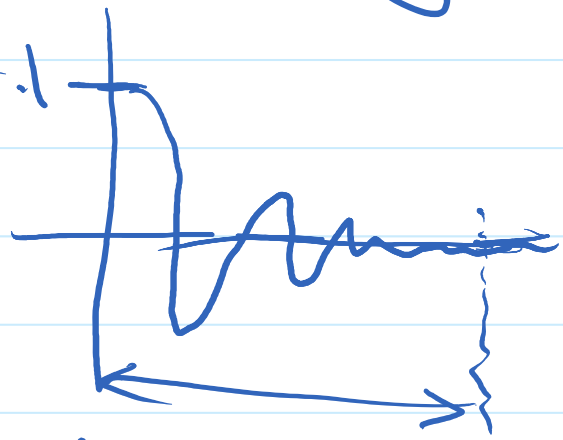


$u = -Kx$

$= -\begin{bmatrix} 22 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ←

if IC $x_0 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$ ←



$T_s = 1$ sec

in matlab

desired eigen values

$P = [-4-3j \quad -4+3j]$

$K = place(A, B, P)$

$K = acker(A, B, P)$

Pole Placement - 2nd Method (Guass Approach)

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This method has two Cases:

Case ①: State space Rep. in First Companion (Caninical form)

$$\dot{X} = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & 0 & 1 \\ -a_n & \cdots & \cdots & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

① check Controllability

② Find Closed loop dynamic matrix

$\dot{x} = (A - BK)x \Leftarrow$, $u = -Kx$, $K = [k_1 \cdots k_n]$ ← gain matrix

$(A - BK) = A_{CL}$

$$A_{CL} = \begin{bmatrix} 0 & 1 & & & 0 \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & \cdots & k_n \end{bmatrix}$$

$$\begin{bmatrix} & 0 & \\ k_1 & \cdots & k_n \end{bmatrix}$$

$$A_{CL} = \begin{bmatrix} 0 & 1 & & & 0 \\ & & 1 & & \\ & 0 & & 0 & 1 \\ (-a_n - k_1) & (-a_{n-1} - k_2) & \cdots & \cdots & (-a_1 - k_n) \end{bmatrix} \quad ①$$

Real closed loop dynamic matrix

Desired Poles: $\mu_1, \ldots, \mu_n$

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + s\alpha_{n-1} + \alpha_n$$

The desired char. Eq will lead to the desired closed loop dynamic Matrix

$$A_{cl,des} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1 \end{bmatrix} \quad (2)$$

desired closed loop dynamic matrix

$$A_{cl} = \begin{bmatrix} 0 & 1 & & & 0 \\ & & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ (-a_n-k_1) & (-a_{n-1}-k_2) & \cdots & & (-a_1-k_n) \end{bmatrix} \quad (1)$$

Real closed loop dynamic matrix

(3) Comparing Eq (1) and Eq (2), we arrive to

$$\left[\begin{array}{l} a_n + k_1 = \alpha_n \\ a_{n-1} + k_2 = \alpha_{n-1} \\ \vdots \\ a_1 + k_n = \alpha_1 \end{array}\right] \Rightarrow \left[\begin{array}{l} k_1 = \alpha_n - a_1 \\ k_2 = \alpha_{n-1} - a_2 \\ \vdots \\ k_n = \alpha_1 - a_1 \end{array}\right] \quad (*)$$

n-Equations

Design Regulator if $T_s = 1\ \text{sec}$, $\zeta = 1$ and $\mu_3 = -30$

Ex:- $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ⇐ First Conical form

① $M = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$ ← rank(3)

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

② $A_{CL} = A - BK$, $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4-k_1 & -3-k_2 & -2-k_3 \end{bmatrix}$$

③ $T_s = 1\ \text{sec}$ $\zeta = 1$

$\Rightarrow T_s = \dfrac{4}{\zeta \omega_n} \Rightarrow \omega_n = 4\ \text{rad/s}$

$\mu_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{1-\zeta^2}\, j$

$= -\omega_n = -4$

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$(s+2)(s+4)(s+30) = s^3 + 38s^2 + 256s + 480$

$s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \alpha_3$

$\alpha_1 = 38$, $\alpha_2 = 256$ $\alpha_3 = 480$

desired closed loop dynamic matrix $A_{CL} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix}$

$a_3 + k_1 = \alpha_3 \Rightarrow 4 + k_1 = 480 \Rightarrow \boxed{k_1 = 476}$

$a_2 + k_2 = \alpha_2 \Rightarrow 3 + k_2 = 256 \Rightarrow \boxed{k_2 = 253}$

$a_1 + k_3 = \alpha_1 \Rightarrow 2 + k_3 = 38 \Rightarrow \boxed{k_3 = 36}$

Case 2: if the Sys is not in the first Companion form

Define $x = T\hat{x}$, T: Similarity transformation matrix

$\dot{x} = T\dot{\hat{x}}$

$\dot{\hat{x}} = T^{-1}\dot{x}$

$\dot{\hat{x}} = T^{-1}(Ax + Bu)$

$\dot{\hat{x}} = \underbrace{T^{-1}AT}_{A}\hat{x} + \underbrace{T^{-1}B}_{B}u$

Design a T such that $T^{-1}AT$ will be in the first Companion form

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$T = M W$

$M = [B \quad AB \; \cdots \; A^{n-1}B]$ Controllability matrix

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdots & \vdots \\ a_1 & 1 & \cdots & \cdots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$\hat{x} = T^{-1} x$

$\dot{\hat{x}} = \bar{A}\hat{x} + \bar{B}u$

$= (T^{-1}AT)\hat{x} + (T^{-1}B)u$

$u = -\hat{k}\hat{x} = -(\hat{k}T^{-1})x = -kx$, where $\hat{k}T^{-1} = k$

Pole Placement Design Steps: Method 2: Bass-Gura Approach

- Check the controllability condition
- Form the characteristic polynomial for *A*
  $|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$
  find a_i's
- Find the Transformation matrix T
- Write the desired characteristic polynomial
  $(s-\mu_1)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$
  and determine the α_i's
- The required state feedback gain matrix is
  $K = [(\alpha_n - a_n) \quad (\alpha_{n-1} - a_{n-1}) \quad \cdots \quad (\alpha_1 - a_1)]\, T^{-1}$

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Design Regulator if the SSR

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ -3 & -2 & -10 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$T_s = 2\,sec \qquad \zeta = 0.8$

①

```
>> M=ctrb(A,B)

M =

     1     1     1
     0     0   -12
     0    -3    27

>> det(M)

ans =

   -36
```

$M = [B \;\; AB \;\; A^2B]$

$\det(M) = -36$

Fully state Controllable

② $|sI - A|$ to fin a_i's

$$\left| \begin{bmatrix} s & & 0 \\ & s & \\ 0 & & s \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ -3 & -2 & -10 \end{bmatrix} \right|$$

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```
>> SIA=s*eye(3)-A

SIA =

[ s - 1,      0,      0]
[     0,  s - 1,     -4]
[     3,      2, s + 10]

>> det(SIA)

ans =

s^3 + 8*s^2 - 11*s + 2
```

$\Rightarrow a_1 = 8, \quad a_2 = -11$

$a_3 = 2$

$s^n + a_1 s^{n-1} \cdots + a_n$

$n = 3$

$T = M\,W$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & & & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \cdots & \vdots \\ a_1 & 1 & \cdots & \cdots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} -11 & 8 & 1 \\ 8 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$a_{n-1} = a_2$, $a_{n-2} = a_1$

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -12 \\ 0 & -3 & 27 \end{bmatrix} \begin{bmatrix} -11 & 8 & 1 \\ 8 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

④ Desired eigenvalue

$T_s = 2\,sec \qquad \zeta = 0.8$

$\omega_n = \dfrac{4}{T_s \zeta} = \dfrac{4}{0.8 \times 2} = \dfrac{4}{1.6} = 2.5\, rad/s$

$\mu_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\, j \qquad \mu_3 = -30$

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$\mu_{1,2} = -2.5 \pm 1.5j$

$\mu_{1,2} = -2.5 \times 0.8 \pm 2.5\sqrt{1-\zeta^2}\, j$

$\mu_3 = -30$ "assumed"

$(s-\mu_1)(s-\mu_2)(s-\mu_3) =$

$= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_3$

$s^3 + 34s^2 + 126.5s + 187.5$

$\alpha_3 = 187.5$

$\alpha_2 = 126.5$

$\alpha_1 = 34$

```
>> T=M*W

T =

    -2     9     1
   -12     0     0
     3    -3     0

>> T_inv=inv(T)

T_inv =

         0   -0.0833         0
         0   -0.0833   -0.3333
    1.0000    0.5833    3.0000
```

(5)

$$K = [(\alpha_n - a_n) \quad (\alpha_{n-1} - a_{n-1}) \quad \cdots \quad (\alpha_1 - a_1)]\, T^{-1}$$

$k = [(187.5-2) \,\vdots\, (126.25-(-11)) \,\vdots\, (34-8)]\, T^{-1}$

$a_1 = 8, \quad a_2 = -11$

$a_3 = 2$