Chapter 7 :- Transcendental functions

- logarithmic
- exponential
- inverse trigonometric

* 7.1 Inverse functions and their derivatives

Def :- a function $f(x)$ is one to one function on the Domain if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ where $x_1, x_2 \in D$

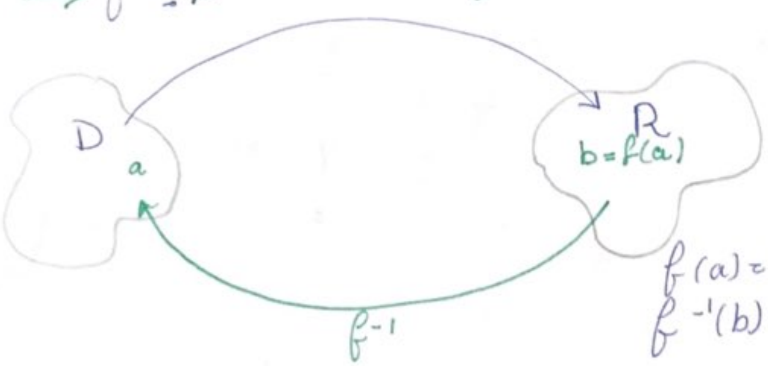
- to know if $f(x)$ is 1-1 function or not we use the horizontal line test

$\Rightarrow$ 1-1 functions intersects 1-1 functions in at most once

Inverse functions

- $D \to R$ is 1-1 function, The inverse function of f is defined by :-

$\Rightarrow f^{-1} : R \to D$ s.t $f^{-1}(b) = a$



$f(a) = b$
$f^{-1}(b) = a$

- To check if the inverse function That you found is right :-

$f^{-1}(f(x)) = x \quad \forall \; x \in D(f)$

↳ if it's Not راحت عليك يا كبير

• How to find $f^{-1}(x)$?

a- Solve for x : احول X موضوع القانون

b- inter change x by y and y by x

c- Replace y by $f^{-1}(x)$

• Theorem :- Given a 1-1 function :

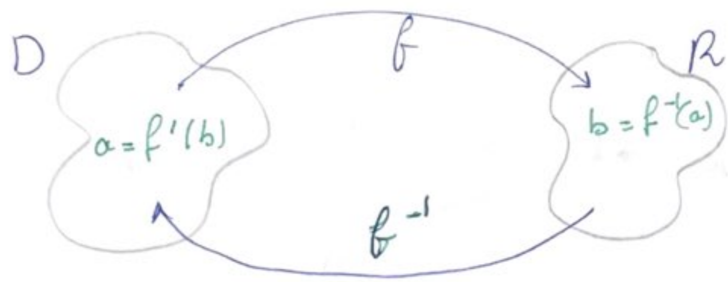
$$f : D \to R$$

with f' exists and never zero on $D(f)$

Then $f^{-1} : R \to D$ is diff s.t on R

where :- $$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$



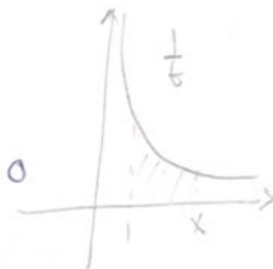
$f : D \to R$

$f' : \bar{D} \to \bar{R}$

$f^{-1} : R \to D$

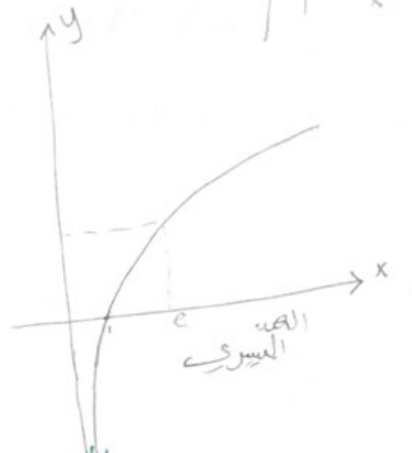
$(f^{-1})' : R \to D$

7.2: Natural logarithms



def:- defined by :- $\ln x = \int_1^x \frac{1}{t}\, dt \quad , \quad x > 0$

D : $(0, \infty)$
R : $(-\infty, \infty)$



Notice that:- $\lim_{x \to \infty} \ln x = 0$

$\lim_{x \to 0^+} \ln x = -\infty$

- Differentiation of Natural logarithm

1- If $g(x)$ is diff and never zero and $y = \ln g(x)$

Then $y' = \frac{dx}{dx} = \frac{g'(x)}{g(x)}$

2- and $\int \frac{g'(x)}{g(x)} = \ln|g(x)| + C$
↳ the number should alway be Positive
That's why we use Absolute value

- Proprietes of Natural logarithm
for $a > 0$ & $b > 0$:-

1- $\ln (a)^*(b) = \ln a + \ln b$
2- $\ln \frac{(a)}{(b)} = \ln a - \ln b$
3- $\ln a^r = r \ln a$
4- $\ln \frac{1}{a} = \ln 1 - \ln a = 0 - \ln a = -\ln a$

* Natural logarithm and trigonometric functions :-

1- $\int \tan x \, dx = \ln|\sec x| + C = -\ln|\cos x| + C$

2- $\int \cot x \, dx = -\ln|\csc x| + C = \ln|\sin x| + C$

3- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

4- $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

Why? (good question :P)

• to prove it :-

1- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int -\frac{(\cos x)'}{(\cos x)} \, dx$

$(\cos x)' = -\sin x$

$= -\ln|\cos x| + C$

$= \ln|\sec x| + C$

VI Remark :- In logarithms $-\ln x = \ln \frac{1}{x}$ (it is proved In the first page)

2- $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{(\sin x)'}{\sin x} \, dx$

$(\sin x)' = (\cos x)$

$= \ln|\sin x| + C$

$= -\ln|\csc x| + C$

* Don't forget the Abs. value :)

3- $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\tan x + \sec x} \, dx = \int \frac{\sec^2 x + \tan x \sec x}{\tan x + \sec x} \, dx$

$(\tan x)' = \sec^2 x$

$(\sec x)' = \tan x \sec x$

$= \int \frac{(\tan x)' + (\sec x)'}{\tan x + \sec x} \, dx$

$= \ln|\tan x + \sec x| + C$

4- $\int \csc x \, dx = \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} \, dx$

$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$

$(\cot x)' = -\csc^2 x$

$(\csc x)' = -\csc x \cot x$

$= \int \frac{-(\cot x)' + -(\csc x)'}{\csc x + \cot x}$

$= -\ln|\csc x + \cot x| + C$

7.3:- Exponensial function

$f(x) = e^x = \ln^{-1} x$

• Remembrer:-

$\ln(1) = 0$

$\ln^{-1} 1 = e$

D: $(-\infty, \infty)$

R: $(0, \infty)$

• Proprieties of e^x :-

1- $e^{\ln x} = x \quad \forall \ x > 0$

2- $\ln e^x = x \quad \forall \ x$

3- $e^{x_1} e^{x_2} = e^{(x_1 + x_2)}$

4- $\frac{e^{x_1}}{e^{x_2}} = e^{(x_1 - x_2)}$

5- $(e^{x_1})^r = e^{r x_1}$

6- $e^{-x} = \frac{1}{e^x}$

• Differentiation and integration of e^x

1- If $y(x) = a^{u(x)}$ → where $a > 0$ and $u(x)$ is diff

Then $y'(x) = a^{u(x)} \ln a \; u'(x)$

2- $\int a^{u(x)} \ln a \; u'(x) \, dx = a^{u(x)} + C$

- Natural logarithm is a special case of General logarithmic function (G.L.f)

General: $y(x) = \log_a u(x) = \frac{\ln u(x)}{\ln a}$ $a>0$ $a \neq 1$ $u(x)>0$

Natural: a (the base) = e so $\ln e = 1 \Rightarrow \log_e u(x) = \frac{\ln u(x)}{1} = \ln u(x)$

- Diff + integration of (G.L.f)

- if $y(x) = \log_a u(x)$ $\xrightarrow{\text{diff}}$ $y'(x) = \frac{1}{\ln a}\left(\frac{u'(x)}{u(x)}\right)$

and $\int \frac{u'(x)}{u(x)} \frac{1}{\ln a} dx = \log_a u(x) + C$

- Proprieties :-

1- $\log_a a^x = x \quad \forall x$ — why? :- $\log_a a^x = \frac{\ln a^x}{\ln a} = x \frac{\cancel{\ln a}}{\cancel{\ln a}} = x$

2- $a^{\log_a x} = x \quad \forall x>0$

3. $\log_a xy = \log_a x + \log_a y$

4. $\log_a \frac{x}{y} = \log_a x - \log_a y$

5. $\log_a \frac{1}{y} = \log_a 1 - \log_a y = 0 - \log_a y = -\log_a y$

6- $\log_a x^y = y \log_a x = y\left(\frac{\ln x}{\ln a}\right)$

7.5 Indeterminante forms & l'hopital Rule

$\frac{0}{0}, \frac{\infty}{\infty}$ → Indeterminante Ratios (we use l'hopital Rule)

$(0)(\infty)$, $\infty - \infty$ → Indeterminante Product

0^0, 1^∞, ∞^0 → Indeterminante Powers

$f(x), g(x)$ - cont - diff

$f(c) = 0$

$g(c) = 0$

Then $\lim_{x\to c} \frac{f}{g} = \frac{0}{0}$

exists & $= \lim_{x\to c} \frac{f'(x)}{g'(x)}$, $g'(x) \neq 0$

VI Remark:- if $f(x)$ is cont Then $\lim_{x\to c} f(x) = \lim_{x\to c} e^{\ln f(x)}$

and $= e^{\ln \lim_{x\to c} f(x)}$ because $\ln x$ is a cont function

- We use This in the ind. powers