# 3.1 Vector Spaces "linear Spaces"

**Def** Let $V$ be a set on ~~with~~ which the operations of

- addition and $\forall \vec{x}, \vec{y} \in V \Rightarrow \vec{x}+\vec{y}$ is a unique element in $V$
- scalar multiplication are defined. $\forall \vec{x} \in V \Rightarrow \alpha \vec{x} \in V$

The set $V$ together with these two operations form a vector space if the following axioms are satisfied:

A1. $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ for any $\vec{x}, \vec{y} \in V$.

A2. $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$ for any $\vec{x}, \vec{y}, \vec{z} \in V$.

A3. $\exists$ an element $\vec{0} \in V$ s.t $\vec{x}+\vec{0} = \vec{x}$ for each $\vec{x} \in V$. zero vector

A4. For each $\vec{x} \in V$, $\exists$ an element $-\vec{x}$ s.t $\vec{x}+(-\vec{x}) = \vec{0}$. Additive inverse

A5. $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$ for each scalar $\alpha \in \mathbb{R}$ and $\vec{x}, \vec{y} \in V$.

A6. $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$ for any scalars $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ and $\vec{x} \in V$.

A7. $(\alpha\beta)\vec{x} = \alpha(\beta\vec{x})$ for any scalars $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ and $\vec{x} \in V$.

A8. $1 \cdot \vec{x} = \vec{x}$ for all $\vec{x} \in V$.

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Note that

1. Any $V \neq \emptyset$ with the operations of addition and scalar multiplication and satisfies the above 8 axioms is called a vector space and its elements are called vectors.

2. Vector spaces are usually denoted by $V, W, S, T$, and its elements "vectors" are denoted by $\vec{u}, \vec{v}, \vec{w}, \vec{x}, \vec{y}, \vec{z}, \ldots$

3. An important component of the definition above is the closure properties of the two operations:
   1. If $\vec{x} \in V$ and $\alpha$ is a scalar, then $\alpha\vec{x} \in V$
   2. If $\vec{x}, \vec{y} \in V$, then $\vec{x}+\vec{y} \in V$.

<u>Exp</u>* Show that $\mathbb{R}^{m\times n}$ together with the usual addition and scalar multiplication of matrices is a vector space.

Let $A=(a_{ij})$ , $B=(b_{ij})$ , $C=(c_{ij})$ be arbitrary elements of $\mathbb{R}^{m\times n}$, $i=1,2,\dots,m$ and $j=1,2,\dots,n$.

A1. $A+B=B+A$ since $a_{ij}+b_{ij}=b_{ij}+a_{ij}$ $\forall i,j$

A2. $(A+B)+C=A+(B+C)$ since $(a_{ij}+b_{ij})+c_{ij}=a_{ij}+(b_{ij}+c_{ij})$ $\forall i,j$

A3. Let $O$ be the $m\times n$ matrix whose entries are all $0$.
$A+O=A$ since $a_{ij}+0=a_{ij}$ $\forall i,j$

A4. Let $-A$ be the $m\times n$ matrix whose $ij^{th}$ entry is $-a_{ij}$.
$A+(-A)=O$ since $a_{ij}+(-a_{ij})=0$ $\forall i,j$

A5. $\alpha(A+B)=\alpha A+\alpha B$ since $\alpha(a_{ij}+b_{ij})=\alpha a_{ij}+\alpha b_{ij}$ $\forall i,j$

A6. $(\alpha+\beta)A=\alpha A+\beta A$ since $(\alpha+\beta)a_{ij}=\alpha a_{ij}+\beta a_{ij}$ $\forall i,j$

A7. $(\alpha\beta)A=\alpha(\beta A)$ since $(\alpha\beta)a_{ij}=\alpha(\beta a_{ij})$ $\forall i,j$

A8. $1A=A$ since $1\,a_{ij}=a_{ij}$ $\forall i,j$.

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<u>Exp</u>** Let $C[a,b]$ denote the set of all real-valued functions defined and continuous on the closed interval $[a,b]$.

Show that $C[a,b]$ together with the usual addition and scalar multiplication of functions is a vector space

Let $f,g,h$ be arbitrary elements of $C[a,b]$ 

A1. $f+g=g+f$ since $\forall x\in[a,b]$
$(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x)$

A2. $(f+g)+h=f+(g+h)$ since $\forall x\in[a,b]$
$[(f+g)+h](x)=(f+g)(x)+h(x)=f(x)+g(x)+h(x)$
$=f(x)+(g+h)(x)=[f+(g+h)](x)$

A3. Let $z(x) = 0 \quad \forall\, x \in [a,b]$. Then,

$f + z = f$ since $\forall x \in [a,b]$

$(f+z)(x) = f(x) + z(x) = f(x) + 0 = f(x)$

A4. Let $(-f)(x) = -f(x) \quad \forall\, x \in [a,b]$. Then

$f + (-f) = z$ since $\forall x \in [a,b]$,

$(f + (-f))(x) = f(x) - f(x) = 0$

A5. $\alpha(f+g) = \alpha f + \alpha g$ since $\forall\, x \in [a,b]$

$[\alpha(f+g)](x) = \alpha f(x) + \alpha g(x) = (\alpha f)(x) + (\alpha g)(x)$

A6. $(\alpha+\beta) f = \alpha f + \beta f$ since $\forall\, x \in [a,b]$

$[(\alpha+\beta)f](x) = (\alpha+\beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x)$

A7. $(\alpha\beta) f = \alpha(\beta f)$ since $\forall\, x \in [a,b]$

$[(\alpha\beta)f](x) = \alpha\beta f(x) = \alpha[\beta f(x)] = \alpha[(\beta f)(x)]$

A8. $1 f = f$ since $\forall\, x \in [a,b]$ $1 f(x) = f(x)$

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Exp (The vector space $P_n$)

- Let $P_n$ denote the set of polynomials of degree less than $n$.
- Define the addition and scalar multiplication operations by

$(p+q)(x) = p(x) + q(x)$ $\qquad P_3 = \{ax^2 + bx + c : a,b,c \in \mathbb{R},\ a \neq 0\}$

$(\alpha p)(x) = \alpha\, p(x)$ for all $x \in \mathbb{R}$.

- $(P_n, +, \cdot)$ is a vector space.

Proof Exactly the same as in Exp$^{**}$

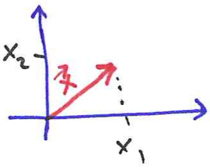
**Exp** The Euclidean n-space $\mathbb{R}^n$, $n=1,2,\dots$ is a vector space (with the usual addition and scalar multiplication).

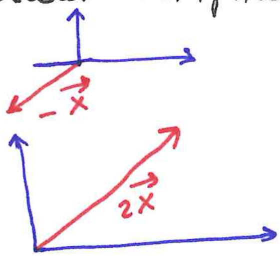
**Proof** Similar to Exp*

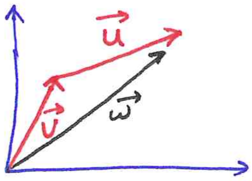
- In case $\mathbb{R}^2 \Rightarrow \vec{x} = (x_1, x_2)$

$\Rightarrow$ length of $\vec{x}$ is $\sqrt{x_1^2 + x_2^2}$

$\Rightarrow \vec{x} + \vec{u} = \vec{w}$

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**Th 3.1.1** If $V$ is a vector space and $\vec{x}$ is any element in $V$, then

1. $0\vec{x} = \vec{0}$
2. $\vec{x} + \vec{y} = \vec{0}$ implies that $\vec{y} = -\vec{x}$ "additive inverse of $\vec{x}$ is unique"
3. $(-1)\vec{x} = -\vec{x}$

**Proof** $\boxed{1}$

$$\vec{0} \underset{(A4)}{=} -\vec{x} + \vec{x} = -\vec{x} + 1\vec{x} \underset{(A8)}{=} -\vec{x} + (1+0)\vec{x} = -\vec{x} + (\vec{x} + 0\vec{x}) \quad (A6)$$

$$= (-\vec{x} + \vec{x}) + 0\vec{x} \quad (A2)$$

$$= \vec{0} + 0\vec{x} \quad (A4)$$

$$= 0\vec{x} \quad (A3)$$

$\boxed{2}$ Let $\vec{x} + \vec{y} = \vec{0}$

$$-\vec{x} = -\vec{x} + \vec{0} \quad (A3)$$

$$= -\vec{x} + \vec{x} + \vec{y}$$

$$= (-\vec{x} + \vec{x}) + \vec{y} \quad (A2)$$

$$= \vec{0} + \vec{y} \quad (A4)$$

$$= \vec{y} \quad (A3)$$

$\boxed{3}$ $\vec{0} = 0\vec{x}$ part $\boxed{1}$

$$= (1 + (-1))\vec{x}$$

$$\vec{0} = 1\vec{x} + (-1)\vec{x} \quad (A6)$$

$$(-1)\vec{x} = -\vec{x} \quad \text{part } \boxed{2}$$

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**Exp** The set $\mathbb{Q}$ together with the usual addition and scalar multiplication is not a vector space.
since $\sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Q}$.

**Exp** $\mathbb{Z}$ is not vector space since $\sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Z}$

**Exp** The set $\mathbb{Z}_4$ is not vector space under $x \bmod 4 = y$ where $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_4 = \{0,1,2,3\}$
since $\frac{1}{2} \times 1 = \frac{1}{2} \notin \mathbb{Z}_4$.