Saturday, May 07, 2022
10:43 PM

8.3.2. Let $n \in \mathbf{N}$, let $\mathbf{a} \in \mathbf{R}^n$, let $s, r \in \mathbf{R}$ with $s < r$, and set

$s < \|w-a\| < r$

$$V = \{\mathbf{x} \in \mathbf{R}^n : s < \|\mathbf{x} - \mathbf{a}\| < r\} \quad \text{and} \quad E = \{\mathbf{x} \in \mathbf{R}^n : s \le \|\mathbf{x} - \mathbf{a}\| \le r\}.$$

Prove that V is open and E is closed. $\forall\, y \in V,\ \exists\, \varepsilon > 0$ $B_\varepsilon(y) \subseteq V$

$w \in B_\varepsilon(y) \Rightarrow \|w-y\| < \varepsilon$ $\qquad \varepsilon < r - \|y-a\|$

$\|w-a\| \le \|w-y\| + \|y-a\| < \varepsilon + \|y-a\|$

$< r - \|y-a\| + \|y-a\| = r$

A $\qquad B$

$\|x-a\| > s$ and $\|x-a\| < r$??

$A \cap B$

8.3.2. Let $y \in V = \{x \in \mathbf{R}^n : s < \|x-a\| < r\}$ and let $\epsilon < \min\{\|y-a\| - s, r - \|y-a\|\}$. If $w \in B_\epsilon(y)$, then

$$\|w-a\| \le \|w-y\| + \|y-a\| < r - \|y-a\| + \|y-a\| = r$$

$\|y-a\| - ?$

and $\quad \|w-a\| \ge \|y-a\| - \|w-y\| \ge \|y-a\| - \varepsilon > s$

$$\|w-a\| \ge \|y-a\| - \|w-y\| > \|y-a\| + s - \|y-a\| = s.$$

$\varepsilon < \varepsilon - \|y-a\|$

Hence $w \in V$ and V is open by definition. $\quad s < \|w-a\| < r \Rightarrow w \in V \Rightarrow B_\varepsilon(y) \subseteq V$

A similar argument shows that $\{x \in \mathbf{R}^n : \|x-a\| > r\}$ and $\{x \in \mathbf{R}^n : \|x-a\| < s\}$ are both open, hence

$$E := \{x \in \mathbf{R}^n : s \le \rho(x,a) \le r\} = \{x \in \mathbf{R}^n : \|x-a\| > r\}^c \cap \{x \in \mathbf{R}^n : \|x-a\| < s\}^c$$

is closed.

$\|w-a\| = \|w-y+y-a\|$

$= \|(y-a) - (y-w)\|$

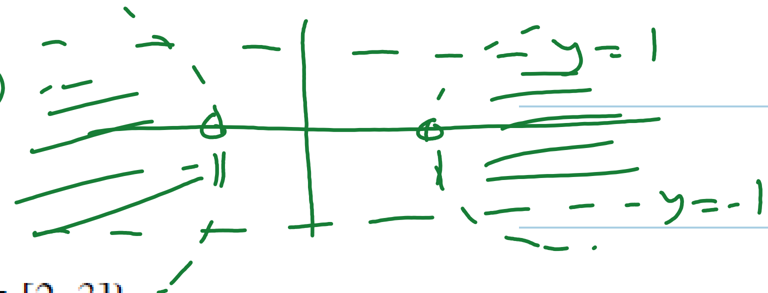
$\ge \|y-a\| - \|w-y\|$

$w \in B_\varepsilon(y) \quad \|y-w\| < \varepsilon$

$-\|y-w\| > -\varepsilon$

8.3.1. Sketch each of the following sets. Identify which of the following sets are open, which are closed, and which are neither. Also discuss the connectivity of each set.

a) $E = \{(x,y) : y \ne 0\}$
b) $E = \{(x,y) : x^2 + 4y^2 \le 1\}$
c) $E = \{(x,y) : y \ge x^2,\ 0 \le y < 1\}$
d) $E = \{(x,y) : x^2 - y^2 > 1,\ -1 < y < 1\}$
e) $E = \{(x,y) : x^2 - 2x + y^2 = 0\} \cup \{(x,0) : x \in [2,3]\}$

(d) 

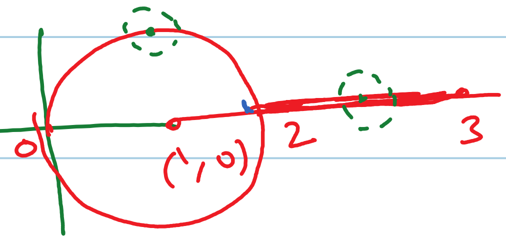
$E^\circ = E$

d) This is the set of points inside the two branches of the hyperbola $x^2 - y^2 = 1$ which lie above the line $y = -1$ and below the line $y = 1$. It is open but not connected.

e) This is the set of points on the circle $(x-1)^2 + y^2 = 1$ or on the x axis between $x = 2$ and $x = 3$. It is closed and connected.

$x^2 - 2x + y^2 = 0 \qquad (x-1)^2 + y^2 = 1$

$\partial E = \bar{E} \setminus E^\circ$
$= E \setminus \phi$
$= E.$



$\bar{E} = E.$

connected.

$E^\circ = \phi$

8.3.3. a) Let $a \leq b$ and $c \leq d$ be real numbers. Sketch a graph of the rectangle

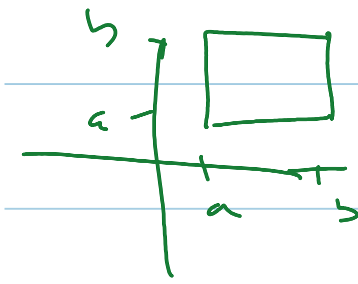
$$[a, b] \times [c, d] := \{(x, y) : x \in [a, b], y \in [c, d]\},$$

and decide whether this set is connected. Explain your answers.

b) Sketch a graph of set

$$B_1(-2, 0) \cup B_1(2, 0) \cup \{(x, 0) : -1 < x < 1\},$$

and decide whether this set is connected. Explain your answers.



8.3.3. a) It is connected (see Remark 9.34 for proof).
b) The set is "dumbbell" shaped. It almost looks connected, except that $(-1, 0)$ and $(1, 0)$ do not belong to the set. Hence a separation can be made, e.g., by using the open sets $V = \{(x, y) : x < -1\}$ and $U = \{(x, y) : x > -1\}$, and applying Remark 8.29.

8.3.9. Show that if E is closed in $\mathbf{R}^n$ and $\mathbf{a} \notin E$, then

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| > 0.$$

E is closed, iff $\forall x_n \in E$, $x_n \to x \Rightarrow x \in E$.

8.3.9. Suppose E is closed and $a \notin E$, but $\inf_{x \in E} \|x - a\| = 0$ $\to \alpha$. Then by the Approximation Property, there exist $x_j \in E$ such that $\|x_j - a\| \to 0$ α, i.e., such that $x_j \to a$. But E is closed, so the limit of the x_j's, namely a, must belong to E, a contradiction.

$\inf_{x \in E} = \alpha$, $\exists\, x_j \in E$ s.t. $x_j \to \alpha = \inf E$

$x_j \in E$ $\quad \inf = 0$

$\|x_j - a\| \to$

$x_j \to a$

$[0, 1]$ $\quad \inf E = 0$

$x_n = \frac{1}{n} \to 0 = \inf.$

$\sup = 1$, $y_n = \frac{n}{n+1} \to 1$

$\left\{ \begin{array}{l} x_j \in E,\ x_j \to a \\ E \text{ closed} \end{array} \right.$

then $\Rightarrow$ $a \in E.$

$(\frac{1}{2},1)\cup(\frac{1}{3},\frac{1}{2})\cup(\frac{1}{4},\frac{1}{3})\cdots$

EXERCISES

8.4.1. Find the interior, closure, and boundary of each of the following subsets of **R**.

a) $E=\{1/n : n\in\mathbf{N}\}$ $=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}\to\{0\}$

b) $E=\bigcup_{n=1}^{\infty}\left(\frac{1}{n+1},\frac{1}{n}\right)$ $\quad \frac{1}{n}\to 0 \not\in E$ $\quad B_\varepsilon(1)\subseteq E$

c) $E=\bigcup_{n=1}^{\infty}(-n,n)$

d) $E=\mathbf{Q}$

8.4.1. a) The closure is $E\cup\{0\}$, the interior is $\emptyset$, the boundary is $E\cup\{0\}$.

E is open $E^\circ=E$, closed $\bar{E}=E$, $\partial E=\bar{E}\setminus E^\circ$

$\partial E=\bar{E}\setminus E^\circ = (E\cup\{0\})\setminus\emptyset = E\cup\{0\}$

b) The closure is $[0,1]$, the interior is E, the boundary is $\{1/n : n\in\mathbf{N}\}\cup\{0\}$.
c) The closure is $\mathbf{R}$, the interior is $\mathbf{R}$, the boundary is $\emptyset$.
d) The closure is $\mathbf{R}$, the interior is $\emptyset$, the boundary is $\mathbf{R}$. $\{0,1,\frac{1}{2},\frac{1}{3},\dots\}$

8.4.10. Let A and B be subsets of $\mathbf{R}^n$.

a) Show that $\partial(A\cap B)\cap(A^c\cup(\partial B)^c)\subseteq\partial A$.
b) Show that if $\mathbf{x}\in\partial(A\cap B)$ and $\mathbf{x}\notin(A\cap\partial B)\cup(B\cap\partial A)$, then $\mathbf{x}\in\partial A\cap\partial B$.
c) Prove that $\partial(A\cap B)\subseteq(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B)$.
d) Show that even in $\mathbf{R}$, there exist sets A and B such that $\partial(A\cap B)\neq(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B)$.

$A=(0,1)$ $\quad A^c=(-\infty,0]\cup[1,\infty)$ $\quad A\cap B\subseteq A$

8.4.10. a) Let $x\in\partial(A\cap B)\cap(A^c\cup(\partial B)^c)$. $\Rightarrow x\in\partial(A\cap B)$ and $x\in A^c\cup(\partial B)^c$

Case 1. $x\in A^c$. Since $B_r(x)$ intersects A, it follows that $x\in\partial A$. $\quad x\in\partial A$: $x\in A^c \Rightarrow x\in\partial A$

$x\notin\partial B$ $\quad x\notin A$ $\quad B_r(x)\cap A\neq\emptyset$, $B_r(x)\cap A^c\neq\emptyset$

$\cdot\ x\in(\partial B)^c \Rightarrow x\in\partial A$

Case 2. $x\in(\partial B)^c$. Since $B_r(x)$ intersects B, it follows that $B_r(x)\subseteq B$ for small $r>0$. Since $B_r(x)$ also intersects $A^c\cup B^c$, it must be the case that $B_r(x)$ intersects A^c. In particular, $x\in\partial A$.

b) Suppose $x\in\partial(A\cap B)$; i.e., suppose $B_r(x)$ intersects $A\cap B$ and $(A\cap B)^c$ for all $r>0$. If $x\notin(A\cap\partial B)\cup(B\cap\partial A)$, then $x\in(A^c\cup(\partial B)^c)\cap(B^c\cup(\partial A)^c)$. But by part a), $A^c\cup(\partial B)^c\subseteq\partial A$ and $B^c\cup(\partial A)^c\subseteq\partial B$. Hence the intersection is a subset of $\partial A\cap\partial B$.

c) Suppose $x\in\partial(A\cap B)$. If $x\in(A\cap\partial B)\cup(B\cap\partial A)$, then there is nothing to prove. If $x\notin(A\cap\partial B)\cup(B\cap\partial A)$, then by part b), $x\in\partial A\cap\partial B$. Hence $x\in(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B)$.

d) If $A=(0,1)$ and $B=[1,2]$, then $\partial(A\cap B)=\emptyset\neq\{1\}=\partial A\cap\partial B\subseteq(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B)$.

$A\cap B=\emptyset$ $\quad \partial(A\cap B)=\emptyset$

$(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B)=\{1\}$

$\emptyset\cup\{1\}\cup(\{0,1\}\cap\{1,2\})$ $\quad \{1\}$

Saturday, May 07, 2022
10:43 PM

7.1.11. Let f_n be integrable on $[0, 1]$ and $f_n \to f$ uniformly on $[0, 1]$. Show that if $b_n \uparrow 1$ as $n \to \infty$, then

$|b_n - 1| < \varepsilon, \forall n \geq N$

$$\lim_{n\to\infty} \int_0^{b_n} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

$\left| \int_0^{b_n} f_n(x)\,dx - \int_0^1 f(x)\,dx \right|$

7.1.11. Since f is integrable, there is an $M > 0$ such that $|f(x)| \le M$ for all $x \in [0,1]$. Choose $n_0 \in \mathbf{N}$ so that $1 - b_{n_0} \le \epsilon/(2M)$ and $N > n_0$ so large that $|f_n(x) - f(x)| < \epsilon/2$ for $n \ge N$ and $x \in [0,1]$. Suppose $n \ge N$ and $x \in [0,1]$. Since the b_n's are increasing, $b_n \le 1$ for all $n \in \mathbf{N}$ and $n \ge n_0$ imply that $1 - b_n \le 1 - b_{n_0}$. Therefore,

$??$

$$\left|\int_0^1 f(x)\,dx - \int_0^{b_n} f_n(x)\,dx\right| \le \int_0^{b_n} |f(x) - f_n(x)|\,dx + \int_{b_n}^1 |f(x)|\,dx$$
$$\le \frac{\epsilon}{2} b_n + M(1 - b_n) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\left|\int f\right| \le \int |f|$

$b_n \to 1$

$b_n \le 1$

$\int_0^{b_n} f + \int_{b_n}^1 f$

$-\int_{b_n}^1$

$\int_0^{b_n} \frac{\varepsilon}{2}\,dx + \int_{b_n}^1 M\,dx$

$b_n \le 1$

$= \frac{\varepsilon}{2} b_n + M(1 - b_n)$

$< \varepsilon/2$

$b_n \uparrow 1$ $\quad b_n \le 1$

$b_n \to 1$

$\le \frac{\varepsilon}{2}(1) + M \frac{\varepsilon}{2M} = \varepsilon.$

$1 - b_n = |b_n - 1| < \frac{\varepsilon}{2M} \; \forall n \ge N$

c) Find an equation of the plane parallel to the hyperplane $x_1 + \cdots + x_n = \pi$ passing through the point $(1, 2, \ldots, n)$.

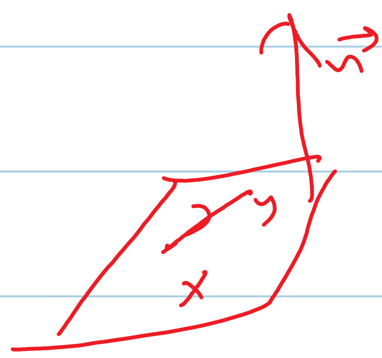
$x_1 \quad x_2 \quad x_n$

$\vec{n} = (1, 1, 1, \ldots, 1)$

$x + 2y + z = 5$

$\vec{n} = (1, 2, 1).$

$\bar{x}(1, 2, \ldots, n), \; y = (x_1, \ldots, x_n)$



$(y - x) \cdot \vec{n} = 0$

$1(x_1 - 1) + 1(x_2 - 2) + \cdots + 1(x_n - n) = 0$

$x_1 + \cdots + x_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$