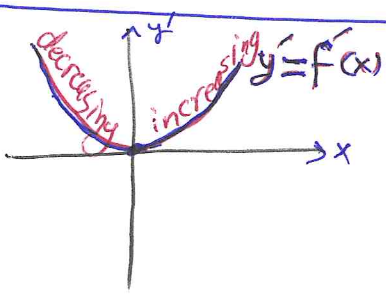
$\boxed{4.4}$ Concavity and Curve Sketching

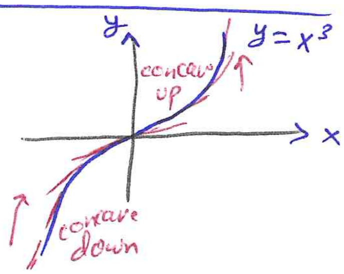
Def Let $f(x)=y$ be a differentiable function on interval I

(a) If f' is increasing on I, then f is concave up on the open interval I

(b) If f' is decreasing on I, then f is concave down on the open interval I.

Example $y=x^3$





The 2nd Derivative Test for Concavity:

* Let $y=f(x)$ be twice-differentiable on an interval I.

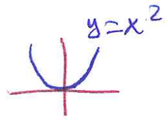
(a) If $f''>0$ on I, then f is concave up on I

(b) If $f''<0$ on I, then f is concave down on I.

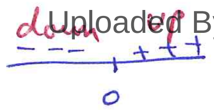
Example: Determine the concavity of

① $y=x^2 \Rightarrow y'=2x \Rightarrow y''=2>0$

$\Rightarrow$ y is concave up on every I.



② $y=x^3 \Rightarrow y'=3x^2 \Rightarrow y''=6x$



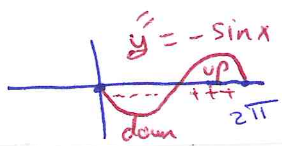
y is concave up on $(0,\infty)$

y is concave down on $(-\infty,0)$

③ $y=3+\sin x$ on $[0,2\pi]$

$y'=\cos x$

$y''=-\sin x$



y concave up on $(\pi,2\pi)$

y concave down on $(0,\pi)$

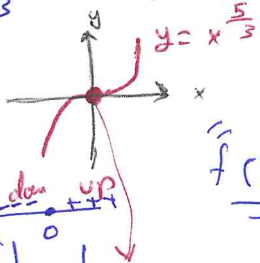
Def A point $x=c$ is called inflection point of the function f if the function f has a tangent line at $x=c$ and changes concavity.

May be horizontal or vertical or oblique

If the function f has an inflection point at $(c, f(c))$ then either $f''(c)=0$ or $f''(c)$ fails to exist (undefined)

Example ① ($f'(c)$ exists but $f''(c)$ fails to exist) Find the inflection point?

$f(x) = x^{\frac{5}{3}}$, $f'(x) = \frac{5}{3} x^{\frac{2}{3}}$, $f''(x) = \frac{10}{9} x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}}$



$f''(0)$ fails to exist

at $x=0$ ① we have a tangent since $f'(0)=0$ (exists)

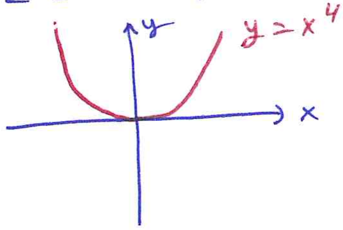
② f changes concavity since $\underset{0}{\overset{\text{down} \quad \text{up}}{--- \cdot +++}}\ y''$

Thus, $(0, f(0)) = (0,0)$ is an inflection point.

Example ② ($f'(c)$ and $f''(c)$ exist)

Find an inflection point of $y = x^4$

$y' = 4x^3$, $y'' = 12x^2 = 0 \iff x = 0$ so $f''(0)=0$

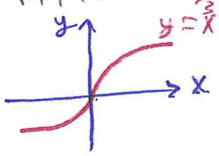


at $x=0$, ① we have a tangent since $f'(0)=0$ (exists)

② f does not change concavity since $\underset{0}{\overset{\text{up} \quad \text{up}}{+++ \cdot +++}}\ y''$

Thus, $(0, f(0)) = (0,0)$ is not inflection point

Example ($f'(c)$ and $f''(c)$ do not exist) Find the inflection point of $y = x^{\frac{1}{3}}$



$y' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$y'' = -\frac{2}{9} x^{-\frac{5}{3}} = \frac{-2}{9\sqrt[3]{x^5}}$

at $x=0$, ① we have a vertical tangent since $f'(0)$ fails to exist.

② f changes concavity at $x=0$ since $\underset{0}{\overset{\text{up} \quad \text{down}}{+++ \cdot ---}}\ y''$ Thus $(0,0)$ is inflection point

Th (2^{nd} Derivative Test for local extrem)

Suppose f'' is continuous on an open interval that contain c.

(1) If $f'(c) = 0$ and $f''(c) < 0$, then f has local max at c.

(2) If $f'(c) = 0$ and $f''(c) > 0$, then f has local min at c.

(3) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails, and the function f may have local max, local min, or neither. (see example ①, ②)

Example ⓐ Find the local extrem of $f(x) = x^3 - 3x + 3$

$f' = 3x^2 - 3 = 0 \iff x = \pm 1$ (critical points) $(1, f(1))$ and $(-1, f(-1))$

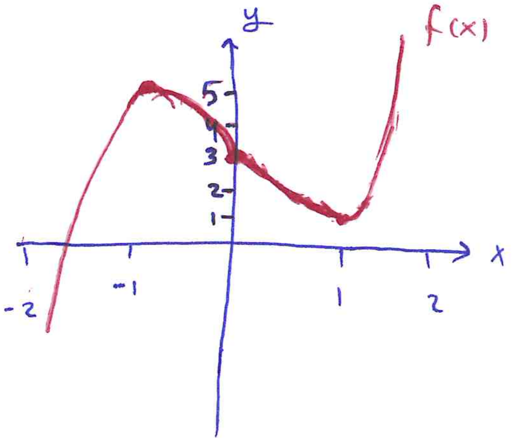
$f''(x) = 6x$

- $f'(1) = 0$ and $f''(1) = 6 > 0 \Rightarrow f$ has local min at $x = 1$
- $f'(-1) = 0$ and $f''(-1) = -6 < 0 \Rightarrow f$ has local max at $x = -1$.

ⓑ Graph the function f

- $f(0) = 3$
- local Max at $(-1, 5)$
- local Min at $(1, 1)$
- $f' = 3x^2 - 3 = 0 \iff x^2 - 1 = 0$

f': $+++$ -1 $----$ 1 $+++$; f



- f is increasing on $(-\infty, -1)$ and $(1, \infty)$
- f is decreasing on $(-1, 1)$

- $f'' = 6x = 0$ down $---$ 0 $+++$ up ; f''

at $x = 0$ we have inflection point $(0, 3)$ since $f(0) = -3$ and so, we have a tangent and f changes concavity.

* To graph a function $y = f(x)$

1) Find the Domain of $f(x)$.
2) Find y' and the critical points.
3) Find where f is increasing and decreasing, local Max & local Min.
4) Find y'' and the inflection points.
5) Find where f is concave up and concave down.
6) Find the asymptotes of f (horizontal and vertical)
7) Plot Key points : x- intercepts and y- intercepts.

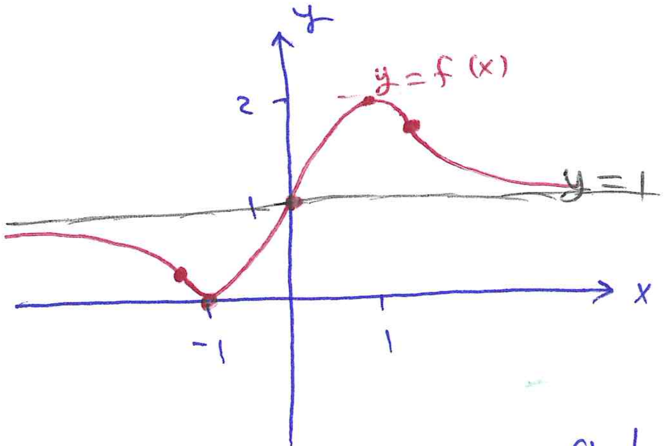
Example: Sketch the graph $f(x) = \dfrac{(x+1)^2}{1+x^2}$

1) $D(f) = (-\infty, \infty)$

2) $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2} = 0 \iff x = \pm 1$, so the critical points are $(1,2)$, $(-1,0)$

3) f': $- - - -$ $\;-1\;$ $+ + +$ $\;1\;$ $- - - -$

- f is increasing on $(-1,1)$
- f is decreasing on $(-\infty,-1)$ and $(1,\infty)$
- f has local min $(-1,0)$
- f has local Max at $(1,2)$



4) $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3} = 0 \iff x = 0, \sqrt{3}, -\sqrt{3}$ are inflection points because f changes concavity

x: $- - - -$ $\;0\;$ $+ + +$

$x^2 - 3$: $+ +$ $\;-\sqrt{3}\;$ $- - - - - -$ $\;\sqrt{3}\;$ $+ +$

5) f'': down $- - -$ $\;-\sqrt{3}\;$ up $+ + +$ $\;0\;$ down $- - -$ $\;\sqrt{3}\;$ up $+ + +$

6) No vertical asymptotes
Horizontal asymptote at $x = 1$
because $\lim_{x \to \infty} f(x) = 1$

7) $(-1,0)$, $(0,1)$