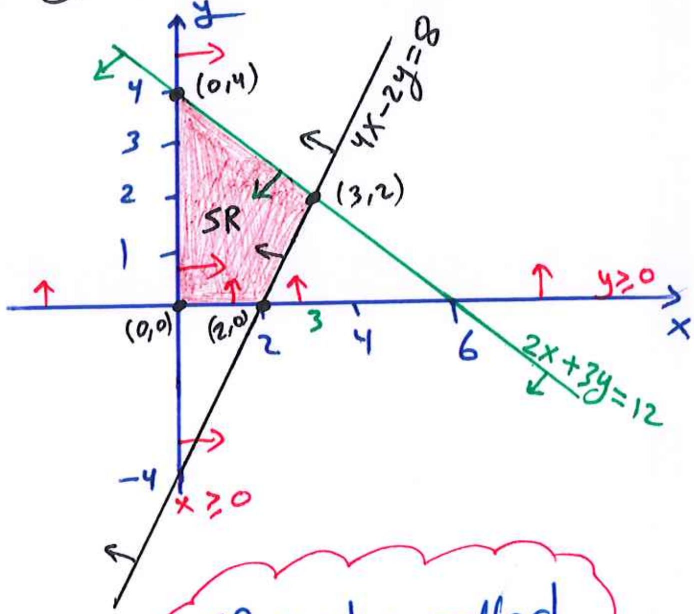
4.2 Linear Programming: Graphical Method

Exp Find the maximum and minimum values of the objective function $f(x,y) = C = 4x + 3y$ in the linear programming problem determined by the constraints:

$$\begin{cases} 2x + 3y \leq 12 \\ 4x - 2y \leq 8 \\ x \geq 0 \ , \ y \geq 0 \end{cases}$$



- Graph the solution Region (SR)

$2x + 3y = 12$

$x = 0 \Rightarrow 3y = 12 \Rightarrow y = 4$

$y = 0 \Rightarrow 2x = 12 \Rightarrow x = 6$

Test Point $(0,0) \Rightarrow 2(0) + 3(0) \leq 12$ ✓

$4x - 2y = 8$

$x = 0 \Rightarrow -2y = 8 \Rightarrow y = -4$

$y = 0 \Rightarrow 4x = 8 \Rightarrow x = 2$

Test point $(0,0) \Rightarrow 4(0) - 2(0) \leq 8$ ✓

SR is also called feasible region

- Note that the feasible region is closed and bounded $\Rightarrow$ the objective function f has max and min values on SR

- Find values of f at the corners

$f(0,0) = 4(0) + 3(0) = 0 + 0 = 0$

$f(0,4) = 4(0) + 3(4) = 0 + 12 = 12$

$f(2,0) = 4(2) + 3(0) = 8 + 0 = 8$

$f(3,2) = 4(3) + 3(2) = 12 + 6 = 18$

To find the point $(3,2) \Rightarrow$ solve

$$\left.\begin{aligned} 4x - 2y &= 8 \quad /{-2} \\ 2x + 3y &= 12 \end{aligned}\right\} \Rightarrow \begin{aligned} \cancel{-2x} + y &= -4 \\ \cancel{2x} + 3y &= 12 \quad + \end{aligned}$$

$4y = 8$

$\boxed{y = 2}$

$2x + 3y = 12$

$2x + 3(2) = 12$

$2x + 6 = 12$

$2x = 6$

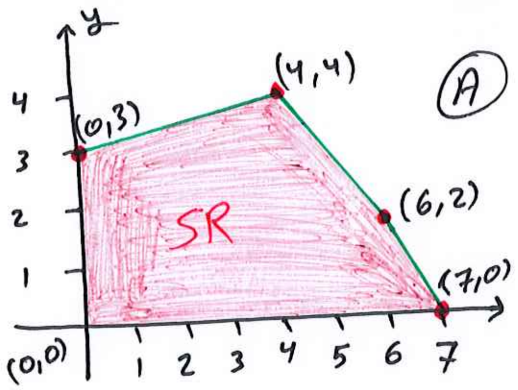
$\boxed{x = 3}$

$(x,y) = (3,2)$ ✓

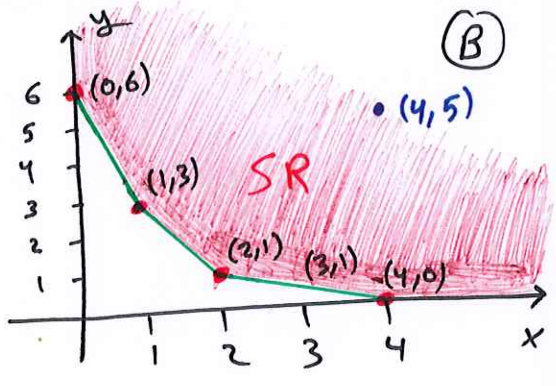
- The Maximum of f is 18 at $(x,y) = (3,2)$

- The Minimum of f is 0 at $(x,y) = (0,0)$

Exp Given the following feasible regions



$C = 9x + 10y$



$f = 4x + 5y$

① Determine if the feasible region bounded and closed or not

The feasible region in (A) is closed and bounded but (B) is not

② Find the maximum and minimum of C in (A) and f in (B)

(A) $C(0,3) = 9(0) + 10(3) = 0 + 30 = 30$

$C(4,4) = 9(4) + 10(4) = 36 + 40 = 76$

$C(6,2) = 9(6) + 10(2) = 54 + 20 = 72$

$C(7,0) = 9(7) + 10(0) = 63 + 0 = 63$

$C(0,0) = 9(0) + 10(0) = 0 + 0 = 0$

The maximum of C is 76 at $(x,y) = (4,4)$

The minimum of C is 0 at $(x,y) = (0,0)$

since the feasible region is closed and bounded

(B) $f(0,6) = 4(0) + 5(6) = 0 + 30 = 30$ → not max

$f(1,3) = 4(1) + 5(3) = 4 + 15 = 19$

$f(2,1) = 4(2) + 5(1) = 8 + 5 = 13$

$f(3,1) = 4(3) + 5(1) = 12 + 5 = 17$

$f(4,0) = 4(4) + 5(0) = 16 + 0 = 16$

when the feasible region is not closed and bounded ⇒ c has max only or min only or no solution

The maximum of f is not found since $f(4,5) = 4(4) + 5(5) = 41 > 30$

The minimum of f is 13 at $(x,y) = (2,1)$

Exp Find the Maximum and Minimum of

$f(x,y) = 2x + 3y$ subject to $\begin{cases} x \geq 1 \\ y \geq 1 \\ y \leq 4 - x \\ y \geq 2x - 2 \end{cases}$

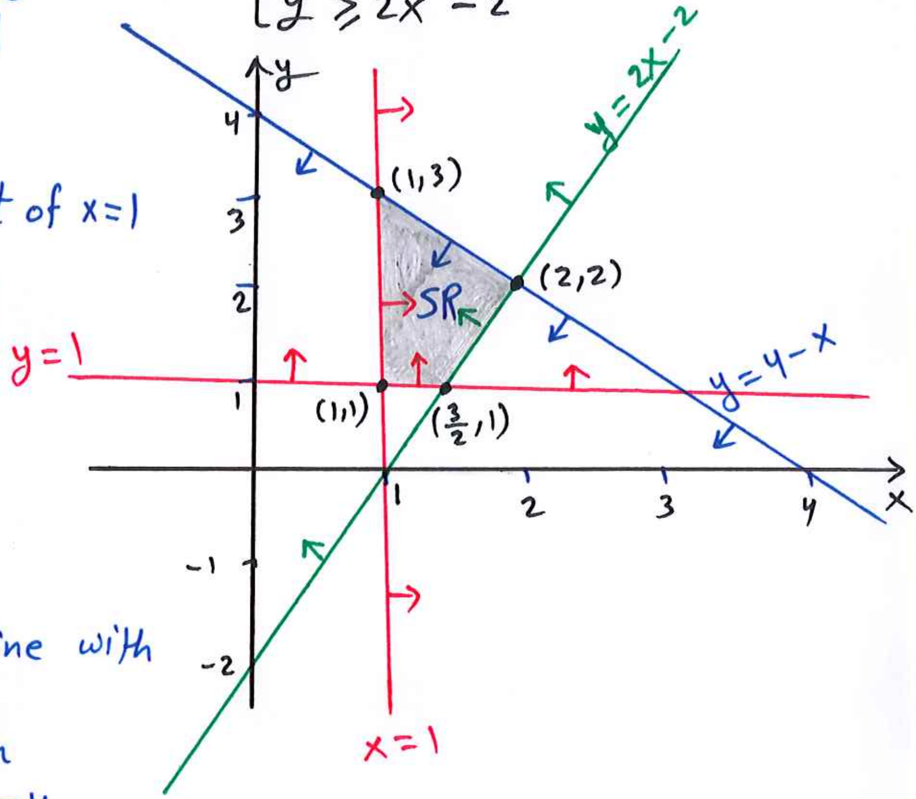
- We sketch the solution region (SR)

$\boxed{x \geq 1} \Rightarrow x = 1$ is vertical line

$\Rightarrow x \geq 1$ all points on the right of $x = 1$

$\boxed{y \geq 1} \Rightarrow y = 1$ is horizontal line

$\Rightarrow y \geq 1$ all points above $y = 1$



$\boxed{y \leq 4 - x} \Rightarrow y = 4 - x$ is line with

- x-intercept $(4,0)$ since when $y = 0 \Rightarrow x = 4$
- y-intercept $(0,4)$ since when $x = 0 \Rightarrow y = 4$
- Test point $(0,0) \Rightarrow 0 \overset{?}{\leq} 4 - 0 \Rightarrow 0 \leq 4$ ✓

$\boxed{y \geq 2x - 2} \Rightarrow y = 2x - 2$ is line with

- x-intercept $(1,0)$ since when $y = 0 \Rightarrow x = 1$
- y-intercept $(0,-2)$ since when $x = 0 \Rightarrow y = -2$
- Test point $(0,0) \Rightarrow 0 \overset{?}{\geq} 2(0) - 2 \Rightarrow 0 \overset{?}{\geq} -2$ ✓

• Then find the corner points

$\underbrace{(1,1)}$, $\underbrace{(\frac{3}{2},1)}$, $\underbrace{(2,2)}$, $\underbrace{(1,3)}$ → intersection $x=1$ with $y=4-x$

$y=4-1$

$y=3$

intersection $y=1$ with $x=1$

intersection $y=1$ with $y=2x-2$

$1=2x-2$

$3=2x$

$x=\frac{3}{2}$

intersection $y=4-x$ with $y=2x-2$

$4-x=2x-2$

$4=3x-2$

$6=3x$

$x=2$

$y=2(2)-2$

$=4-2$

$y=2$

• Evaluate f at the corner points

$$f(x,y)=2x+3y$$

$f(1,1)=2(1)+3(1)=2+3=5$

$f(\frac{3}{2},1)=2(\frac{3}{2})+3(1)=3+3=6$

$f(2,2)=2(2)+3(2)=4+6=10$

$f(1,3)=2(1)+3(3)=2+9=11$

Hence, f has Max of 11 at $(x,y)=(1,3)$

and f has Min of 5 at $(x,y)=(1,1)$

• Note that the SR is closed and bounded ⇒ f has Max and Min points

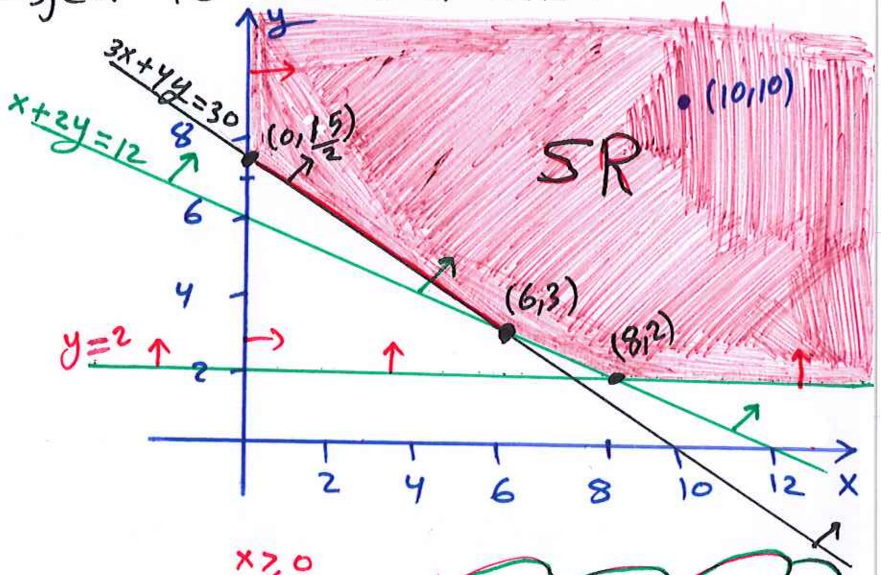
Exp Find maximum and minimum values of the objective function $g = 3x + 4y$ subject to the constraints:

$x + 2y \geq 12$

$3x + 4y \geq 30$

$x \geq 0 , y \geq 2$

- Graph the solution Region (SR)



$x + 2y = 12$

$x = 0 \Rightarrow 2y = 12 \Rightarrow y = 6$

$y = 0 \Rightarrow x = 12$

Test point $(0,0) \Rightarrow 0 + 2(0) \not\geq 12$

$3x + 4y = 30$

$x = 0 \Rightarrow 4y = 30 \Rightarrow y = \frac{15}{2}$

$y = 0 \Rightarrow 3x = 30 \Rightarrow x = 10$

Test point $(0,0) \Rightarrow 3(0) + 4(0) \not\geq 30$

$y = 2 \Rightarrow$ Test point $(0,0) \Rightarrow 0 \not\geq 2$

SR is not bounded
$\Downarrow$
g has only min or only max or no solution

To find the point $(6,3) \Rightarrow$ solve

$3x + 4y = 30$

$x + 2y = 12$ multiply by -3

$\cancel{3x} + 4y = 30$

$\cancel{-3x} - 6y = -36$ +

$-2y = -6$

$\boxed{y = 3}$

$x + 2y = 12$

$x + 6 = 12$

$\boxed{x = 6}$

$x + 2y = 12$

$y = 2 \Rightarrow x + 4 = 12 \Rightarrow x = 8$

- Find values of g at the corner

$g(0, \frac{15}{2}) = 3(0) + 4(\frac{15}{2}) = 30$

$g(6,3) = 3(6) + 4(3) = 30$

$g(8,2) = 3(8) + 4(2) = 32$ → not max

- Hence, g has minimum value at $(x,y) = (0, \frac{15}{2})$ and at $(x,y) = (6,3)$ and at all points between them (on the line joining these two corners)

- g has no maximum since $g(10,10) = 3(10) + 4(10) = 70 > 32$