

Ch.2 :- Limits and Continuity

Q₁ Find the following limits :-

$$(a) \lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2} = \frac{0}{0}$$

$$\lim_{t \rightarrow -1} \frac{(t+1)(t+2)}{(t+1)(t-2)} = \lim_{t \rightarrow -1} \frac{t+2}{t-2} = \frac{1}{-3}$$

$$(b) \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$$

$$(c) \lim_{x \rightarrow 1} \frac{x^4 - 1}{x^3 - 1} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{(x^2 - 1)(x^2 + 1)}{(x - 1)(x^2 + x + 1)} = \lim_{x \rightarrow 1} \frac{(x+1)(x-1)(x^2 + 1)}{(x-1)(x^2 + x + 1)} = \frac{4}{3}$$

$$(d) \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{3\theta} = \frac{1}{3} \cdot 2 \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

OR

$$\lim_{\theta \rightarrow 0} \frac{\frac{\sin 2\theta}{\theta}}{\frac{3\theta}{\theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta}}{\lim_{\theta \rightarrow 0} 3} = \frac{2}{3}$$

$$\textcircled{e} \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin(2\theta)} = \frac{0}{0}$$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin(2\theta)} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} &= \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{2 \sin \theta \cos \theta (1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{2 \cos \theta (1 + \cos \theta)} \\ &= \frac{0}{2(1+1)} = \frac{0}{4} = 0 \end{aligned}$$

OR

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin(2\theta)} = \frac{\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}}{\lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{\theta}} = \frac{0}{2} = 0$$

$$\textcircled{f} \quad \lim_{x \rightarrow 0} \frac{1 + \sqrt{x}}{1 - \sqrt{x}} = \frac{\sqrt{1}}{-\sqrt{1}} = \boxed{-1}$$

$$\textcircled{g} \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 (1 + \frac{1}{x^2})}}{x (1 + \frac{1}{x})}$$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}}{x (1 + \frac{1}{x})}$$

Note:-

$$\sqrt{x^2} = |x|$$

$$(\sqrt{x})^2 = x$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\lim_{x \rightarrow -\infty} \frac{|x| \sqrt{1 + \frac{1}{x^2}}}{x \left(1 + \frac{1}{x}\right)}$$

$$\lim_{x \rightarrow -\infty} \frac{-x \sqrt{1 + \frac{1}{x^2}}}{x \left(1 + \frac{1}{x}\right)} = \boxed{-1}$$

(2) $\lim_{x \rightarrow +\infty} \left(\sqrt{x^2+1} - \sqrt{x^2-x} \right)$ Multiplying by the conjugate

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2+1} - \sqrt{x^2-x} \right) \left(\frac{\sqrt{x^2+1} + \sqrt{x^2-x}}{\sqrt{x^2+1} + \sqrt{x^2-x}} \right)$$

$$\lim_{x \rightarrow \infty} \frac{x+1}{\sqrt{x^2+1} + \sqrt{x^2-x}} = \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{1}{x}\right)}{|x| \sqrt{1 + \frac{1}{x^2}} + |x| \sqrt{1 - \frac{1}{x}}}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{1}{x}\right)}{|x| \left(\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}} \right)}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{1}{x}\right)}{x \left(\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}} \right)}$$

$$= \frac{1}{1 + \sqrt{1}} = \frac{1}{2}$$

$$(h) \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}} = \frac{\sqrt[3]{1}}{\sqrt[3]{1}} = 1$$

$$(j) \lim_{t \rightarrow 3^+} \frac{\lfloor t \rfloor}{t}$$

[H] : Greatest Integer function

$$= \frac{\lfloor 3.1 \rfloor}{3} = \boxed{1}$$

$$(k) \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$$

By Sandwich Th

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-|x| \leq |x| \sin\left(\frac{1}{x}\right) \leq |x|$$

$$0 = \lim_{x \rightarrow 0} -|x| \leq \lim_{x \rightarrow 0} |x| \sin\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0} |x| = 0$$

Then By Sandwich Th. $\lim_{x \rightarrow 0} |x| \sin\left(\frac{1}{x}\right) = 0$

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$$

OR

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$$

$$= \lim_{x \rightarrow 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

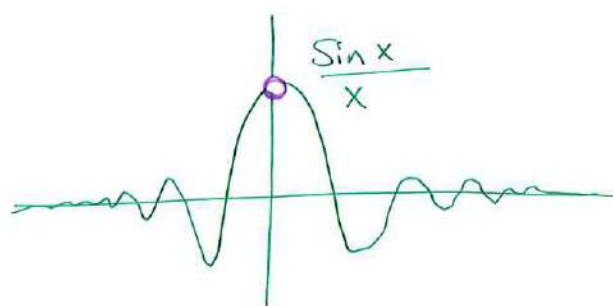
Assume $y = \frac{1}{x}$

$$x \rightarrow 0$$

$$y \rightarrow \infty \text{ or } -\infty$$

$$= \lim_{y \rightarrow \pm\infty} \frac{\sin y}{y}$$

$$= 0$$



Q2 Find the asymptotes of the following functions

Summary :-

(a) $f(x) = \frac{x+1}{x-1}$

Asymptotes :-

Horizontal Asy

$$y = b$$

$$\deg g(x) < \deg h(x)$$

$$\lim_{x \rightarrow \infty} f(x) = b$$

Vertical Asy.

$$x = a$$

Check Zero of Denominator

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty$$

OR

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty$$

Oblique Asy

$$y = ax + b$$

$$\deg g(x) > \deg h(x)$$

Long Division

a) $f(x) = \frac{x+1}{x-1}$, $\boxed{x \neq 1}$

O. Asy: None

H. Asy: Check $\lim_{x \rightarrow \pm\infty} f(x)$

$$\lim_{x \rightarrow \infty} \frac{x+1}{x-1} = 1$$

$$\left\{ \begin{array}{l} \text{H. Asy.} \\ \boxed{y=1} \end{array} \right.$$

OR $\lim_{x \rightarrow -\infty} \frac{x+1}{x-1} = 1$

V. Asy: Check $\lim_{x \rightarrow 1} f(x)$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = \frac{2}{\text{small}^+} = \infty$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = \frac{2}{\text{small}^-} = -\infty$$

V. Asy

for the graph.

- X intercept

$$0 = \frac{x+1}{x-1}$$

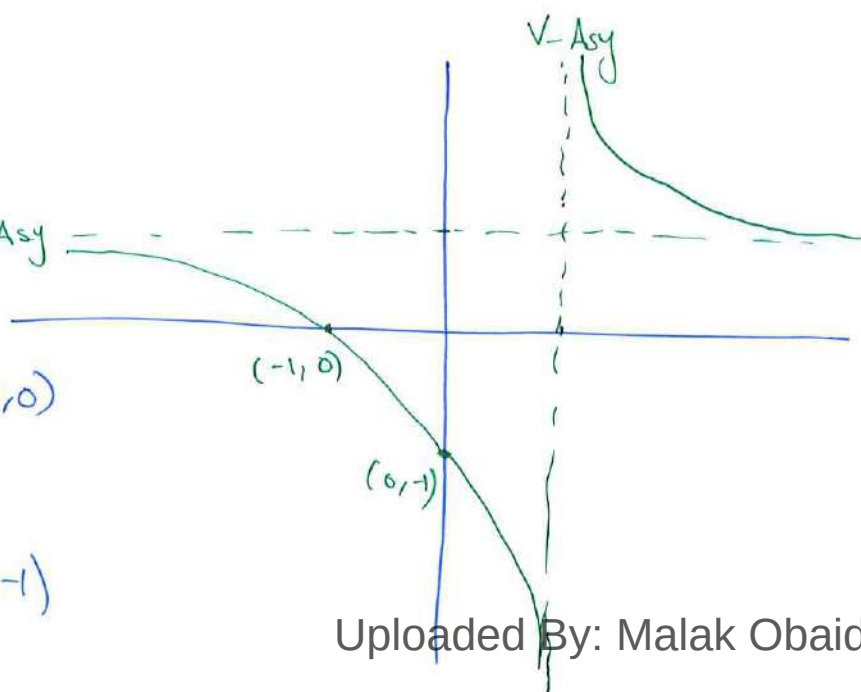
$$0 = x+1 \rightarrow x = -1 \quad (-1, 0)$$

- y-intercept

$$y = \frac{0+1}{0-1}$$

$$y = -1 \quad (0, -1)$$

H. Asy



⑥ $y = \frac{x^3 + 1}{x^2}$, $\boxed{D. \quad x \neq 0}$

H. Asy : None

O. Asy : $\boxed{y = x}$

$$\frac{x^2 \sqrt{\frac{x}{x^3 + 1}}}{-x^3}$$

$$\boxed{\frac{x^3 + 1}{x^2} = x + \frac{1}{x^2}}$$

V. Asy : check $\lim_{x \rightarrow 0} \frac{x^3 + 1}{x^2}$

$$\lim_{x \rightarrow 0^+} \frac{x^3 + 1}{x^2} = \frac{1}{\text{Small}^+} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{x^3 + 1}{x^2} = \frac{1}{\text{Small}^-} = -\infty$$

V. Asy : $\boxed{x = 0}$

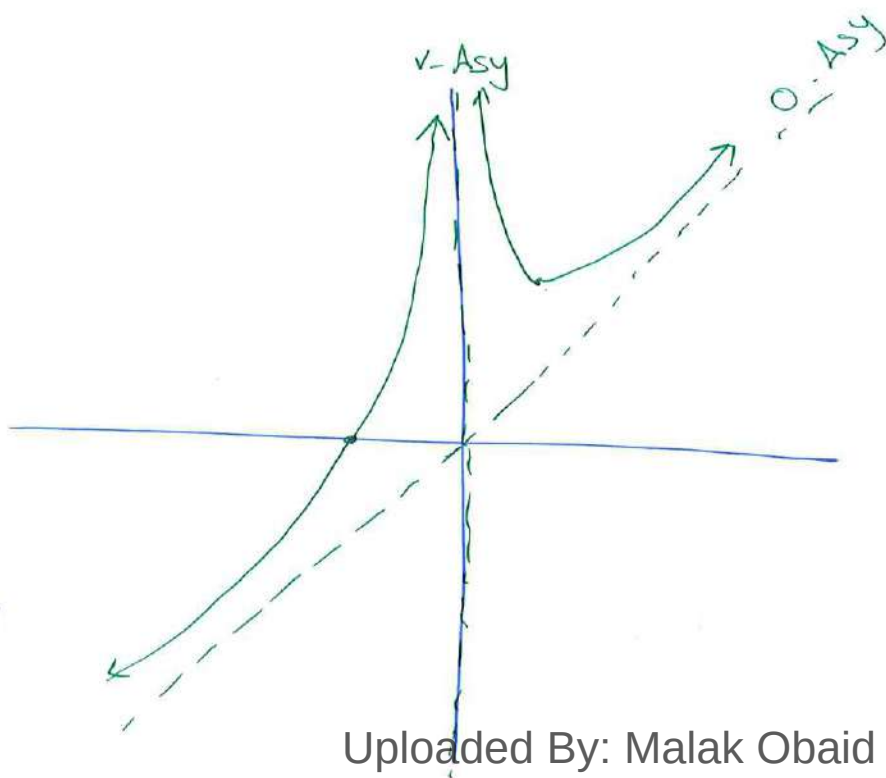
for the graph.

- x-intercept $0 = \frac{x^3 + 1}{x^2}$

$$\begin{aligned} x^3 + 1 &= 0 \\ x^3 &= -1 \rightarrow x = -1 \\ &(-1, 0) \end{aligned}$$

y-intercept $y = \frac{1}{0}$ DNE

No y-intercept.



③ $f(x) = \frac{x^2+1}{x-1}$ D. $x \neq 1$

H. Asy : None

O. Asy $\rightarrow$ $y = x+1$

$$\begin{array}{r} x+1 \\ x+1 \overline{) x^2+1} \\ \underline{-x^2+x} \\ x+1 \\ \underline{-x+1} \\ 2 \end{array}$$

linear

$$\frac{x^2+1}{x-1} = \boxed{x+1} + \frac{2}{x-1}$$

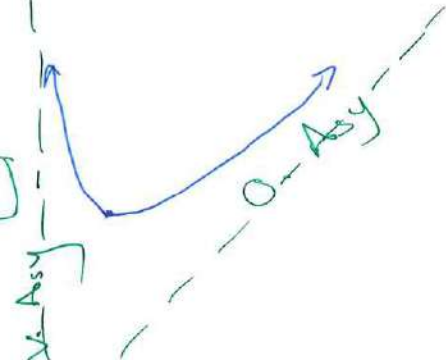
O. Asy

V. Asy : Check $\lim_{x \rightarrow 1} f(x)$

$$\lim_{x \rightarrow 1^+} \frac{x^2+1}{x-1} = \frac{2}{\text{Small}^+} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x^2+1}{x-1} = \frac{2}{\text{Small}^-} = -\infty$$

V. Asy $x=1$



for the graph

x-intercept $0 = \frac{x^2+1}{x-1}$

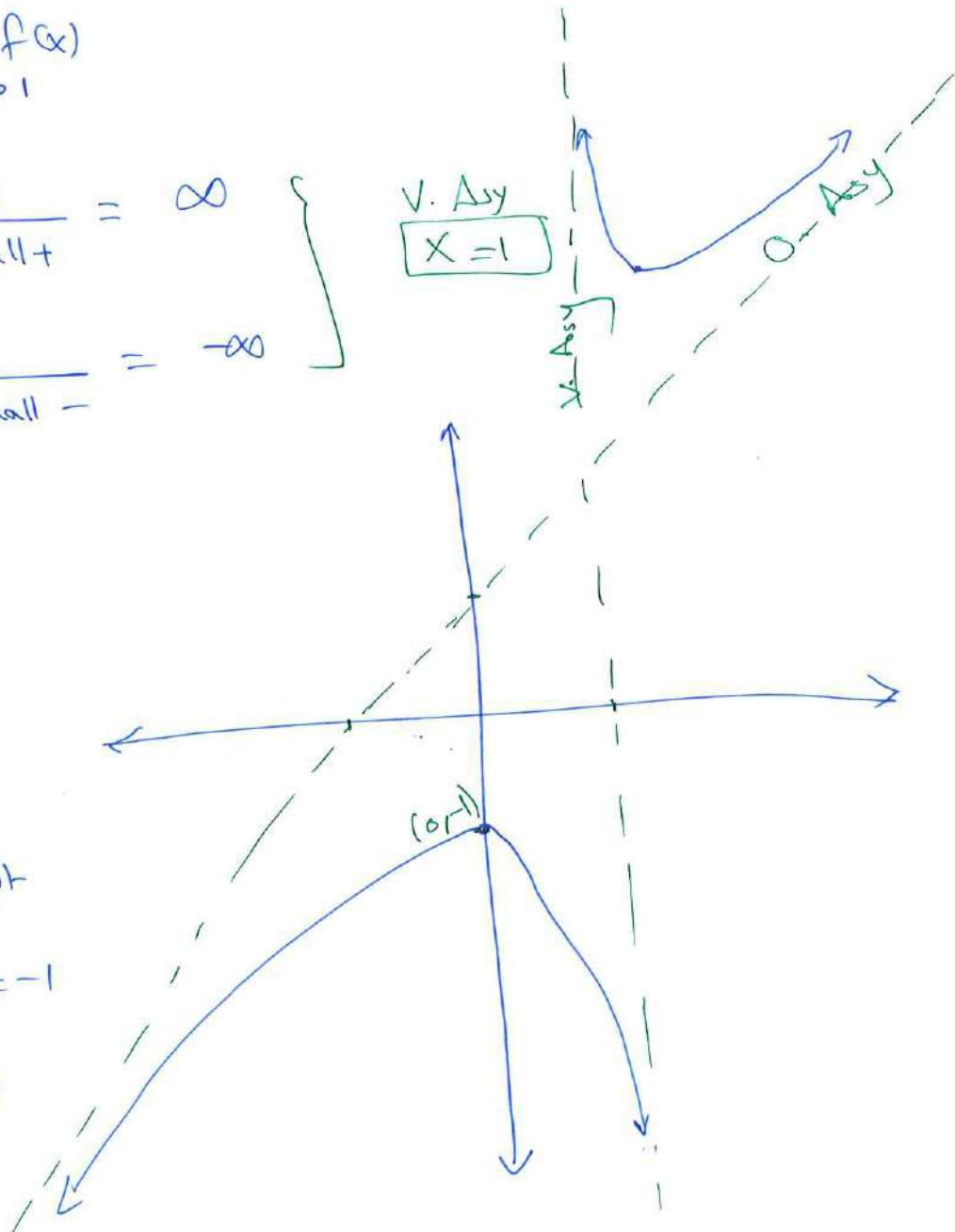
$$x^2+1=0$$

No x-intercept

y-intercept

$$y = \frac{1}{-1} = -1$$

$(0, -1)$



$$d) f(x) = \frac{x^3 + 1}{x^2 - 1}, \quad \boxed{x \neq \pm 1}$$

$$\begin{array}{r} x \\ x^2 - 1 \overline{) x^3 + 1} \\ \underline{-x^3 + x} \\ x + 1 \end{array}$$

• H. Asy : None

• O. Asy : $\boxed{y = x}$

$$\frac{x^3 + 1}{x^2 - 1} = \underbrace{x}_{\text{O. Asy}} + \frac{x+1}{x^2 - 1}$$

• V. Asy : Check $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow -1} f(x)$

$$\lim_{x \rightarrow 1^+} \frac{x^3 + 1}{x^2 - 1} = \frac{2}{\text{Small } +} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x^3 + 1}{x^2 - 1} = \frac{2}{\text{Small } -} = -\infty$$

$$\boxed{\text{V. Asy } x = 1}$$

$$\lim_{x \rightarrow -1} \frac{x^3 + 1}{x^2 - 1} = \frac{0}{0} \rightarrow \boxed{x = -1} \text{ Removable Discontinuity}$$

$$\lim_{x \rightarrow -1} \frac{(x+1)(x^2 + x + 1)}{(x+1)(x-1)} = \frac{3}{-2}$$

for the graph

x-intercept: $0 = \frac{x^3 + 1}{x^2 - 1}$

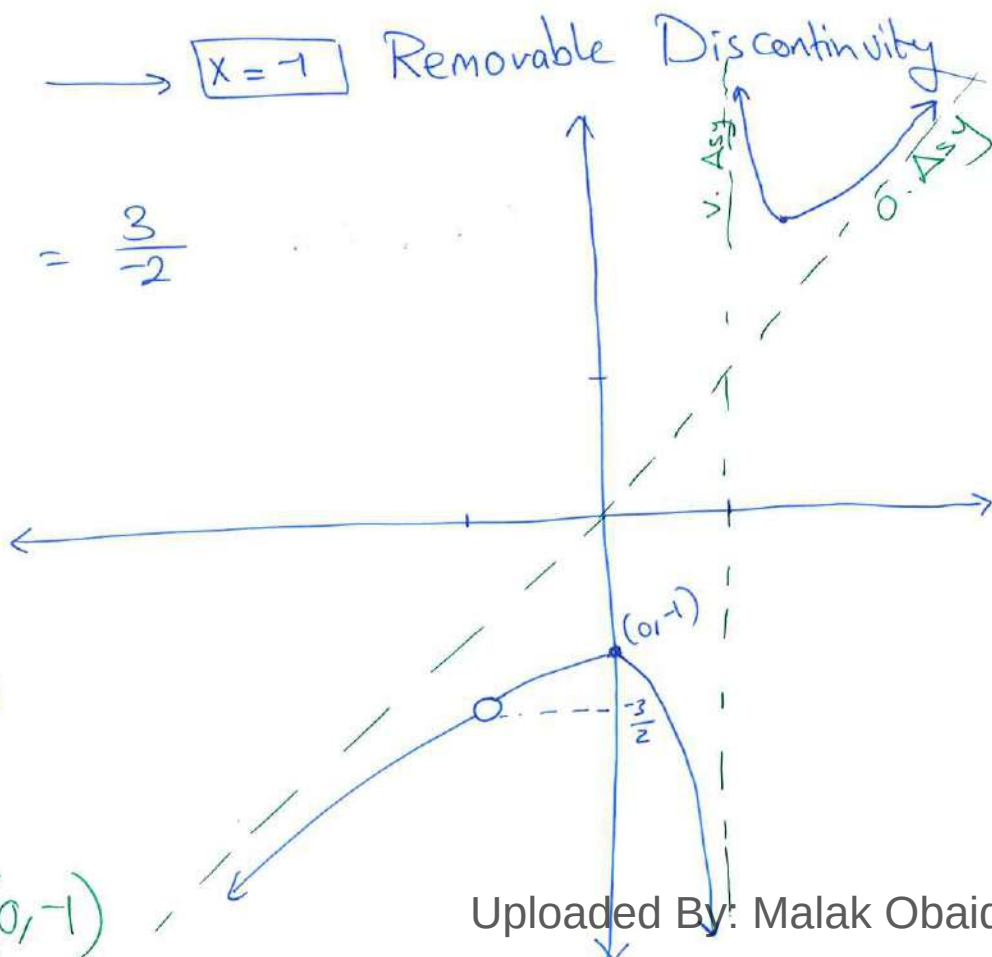
$$x^3 + 1 = 0$$

$$x = -1 \notin D$$

Reject

y-intercept $y = \frac{1}{-1}$

STUDENTS-HUB.com $(0, -1)$



Q₃

Find the value of a, b ??

$$g(x) = \begin{cases} ax + 2b & x \leq 0 \\ x^2 + 3a - b & 0 < x \leq 2 \\ 3x - 5 & x > 2 \end{cases}$$

$g(x)$ is cont. everywhere

Then sketch the graph of the function

* * Since $g(x)$ is cont at $x=0$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^-} g(x)$$

$$0 + 3a - b = 2b \rightarrow \boxed{a = b}$$

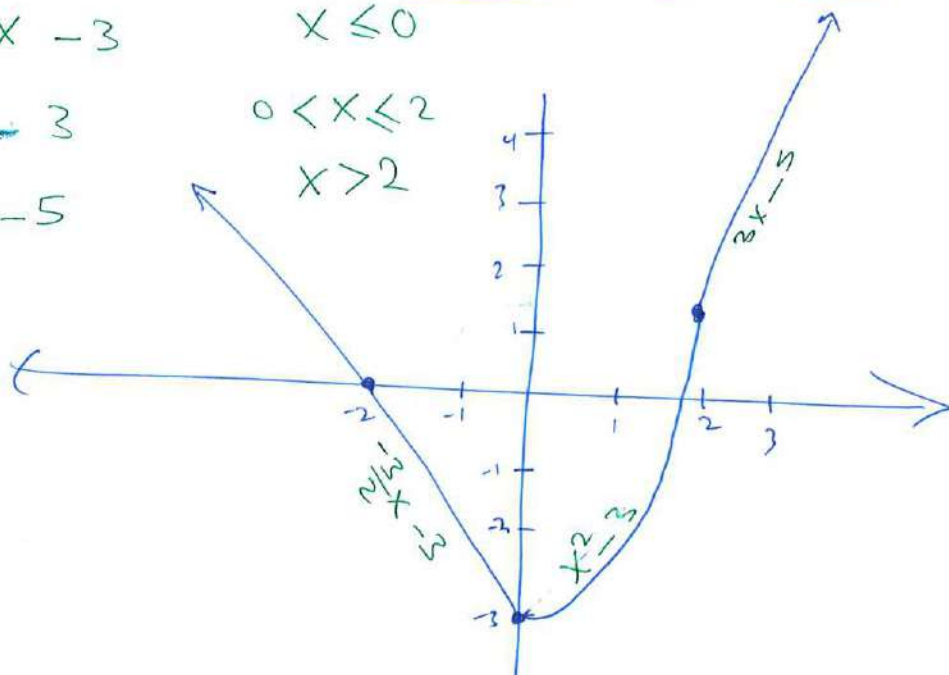
* * Since $g(x)$ is cont at $x=2$

$$\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^-} g(x)$$

$$1 = 4 + 3a - b \rightarrow 3a - a = -3$$

$$\boxed{b = -\frac{3}{2}}, \boxed{a = -\frac{3}{2}}$$

$$g(x) = \begin{cases} -\frac{3}{2}x - 3 & x \leq 0 \\ x^2 - 3 & 0 < x \leq 2 \\ 3x - 5 & x > 2 \end{cases}$$



[Q4]

Continuous Extension

$$h(x) = \frac{x^2 + 3x - 10}{x - 2}, \quad [x \neq 2]$$

$$h(2) = \frac{0}{0} \longrightarrow [x=2] \text{ Removable Discontinuity}$$

$$\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+5)}{x-2} = 7$$

The continuous Extension of $h(x)$ at $x=2$

$$H(x) = \begin{cases} \frac{x^2 + 3x - 10}{x - 2} & x \neq 2 \\ 7 & x = 2 \end{cases}$$

[Q5] Use the Intermediate Value Theorem :- I.V.T
to show that the $f(x) = x^3 + 2x^2 + 2$ has a root (solution)
Conditions:-

① $f(x)$ is cont. for all x because it's a polynomial
 $\rightarrow f(x)$ is cont on $[-2, 2]$

② $y_0 = 0$ lie between $f(-2)$ and $f(2)$

$$f(-2) = -14 \quad f(-2) \leq y_0 \leq f(2)$$

$$f(2) = 2$$

$\rightarrow$ Then by I.V.T, $\exists c \in [-2, 2]$ such that $f(c) = y_0 = 0$

$\rightarrow$ This means that c is a root (solution) of $f(x) = 0$