5.2 Logarithmic Functions and Their Properties

The Logarithmic function is defined by

$$y = \log_a x = \frac{\ln x}{\ln a} \quad , \quad \underbrace{x > 0}_{\text{domain}} \; , \quad a > 0, \; a \neq 1$$

base

$$y = \log_a x \iff a^y = x$$

logarithmic form — exponential form

Exp write the following logarithmic equations using exponential form:

① $4 = \log_2 16 \iff 2^4 = 16$

② $\frac{1}{2} = \log_9 3 \iff 9^{\frac{1}{2}} = 3$

③ $-2 = \log_5 \left(\frac{1}{25}\right) \iff 5^{-2} = \frac{1}{25}$

Natural logarithmic function is $y = \log_e x = \frac{\ln x}{\ln e} = \ln x$

when the base $a = e \Rightarrow \ln e = 1$

- Graph $y = \ln x$, domain $= (0, \infty)$

x	$y = \ln x$
0.05	-3
0.1	-2.3
0.5	-0.7
1	0
2	0.7
3	1.1
10	2.3
e	1

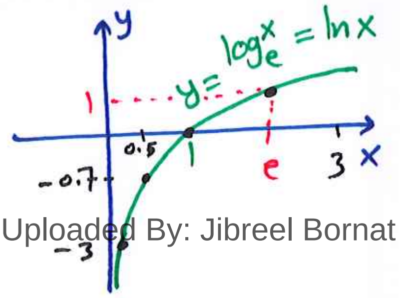
Using Calculator:

$\log x$ means $\log_{10} x$

$\ln x$ means $\log_e x$

Exp $\log 2 \approx 0.301$

$\ln 2 \approx 0.693$



negative y-axis is the vertical asymptote

$e \approx 2.718$

Exp Solve for x

[1] $\log_3 x = 4 \Rightarrow x = 3^4 = 3 \times 3 \times 3 \times 3 = 81$

[2] $\log_{16} x = -\frac{1}{2} \Rightarrow x = (16)^{-\frac{1}{2}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$

[3] $\log(4x+20) = 2 \Rightarrow \log_{10}(4x+20) = 2 \Rightarrow 4x+20 = 10^2$

$\Rightarrow 4x+20 = 100 \Rightarrow 4x = 80 \Rightarrow \boxed{x = 20}$

[4] $\ln(3x-17) = 0 \Rightarrow \log_e(3x-17) = 0 \Rightarrow e^0 = 3x-17$

$\Rightarrow 3x-17 = 1 \Rightarrow 3x = 18 \Rightarrow \boxed{x = 6}$

Exp write the following equation in logarithmic form:

[1] $2^4 = 16 \Leftrightarrow \log_2 16 = 4$

[2] $3^{-1} = \frac{1}{3} \Leftrightarrow \log_3\left(\frac{1}{3}\right) = -1$

Exp Graph $y = \log_2 x$

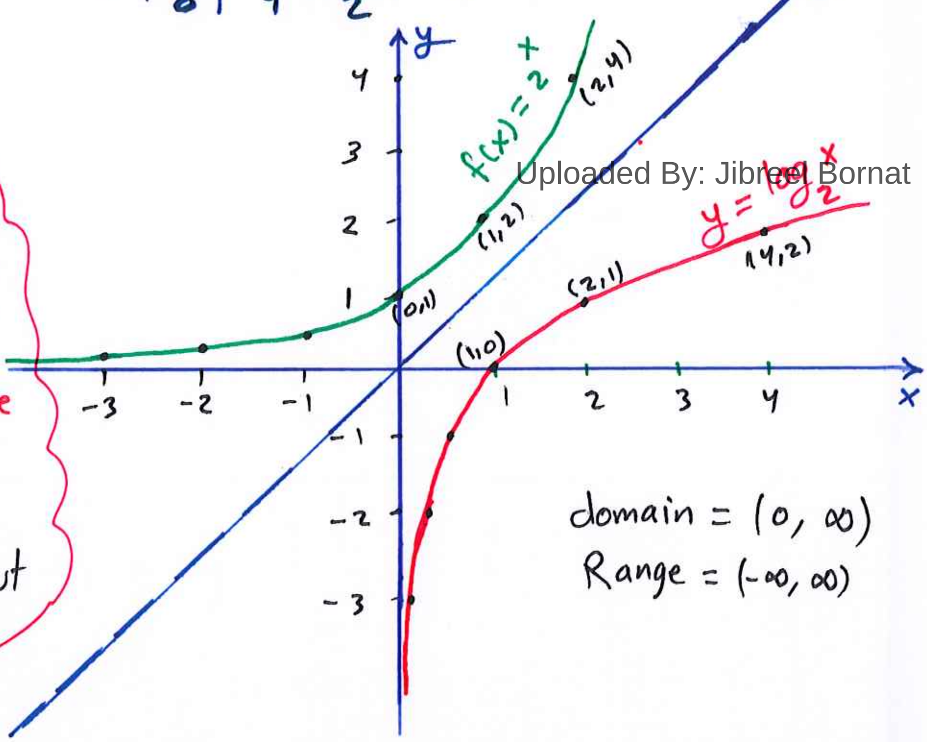
$y = \log_2 x \Leftrightarrow x = 2^y \Rightarrow$

y	-3	-2	-1	0	1	2	3
x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Exp Graph $f(x) = 2^x$

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

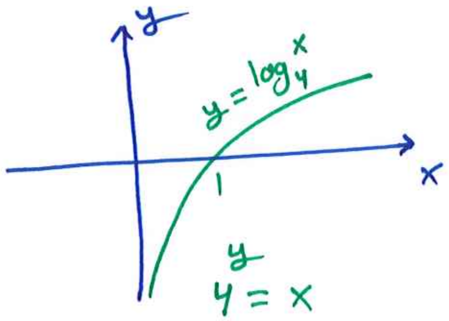
2^x is the inverse function of $\log_2 x$ and $\log_2 x$ is the inverse function of 2^x since they are symmetric about $y = x$



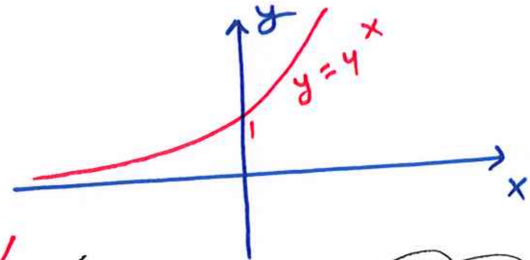
Exp Graph ① $y = \log_4 x$ ② $y = 4^x$

③ $y = \log_9 x$ ④ $y = 9^x$ ⑤ $y = \log_9 (-x)$

①

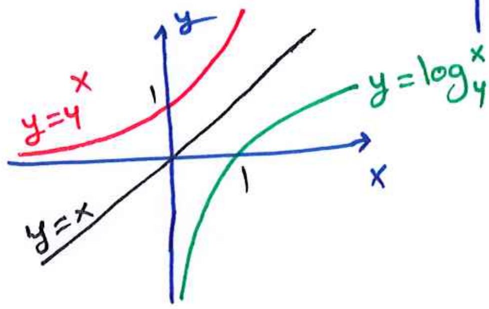


②



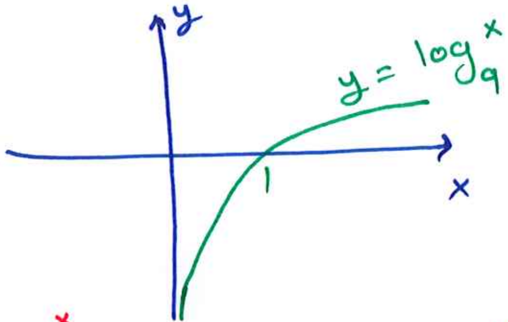
$4^y = x$

y	-1	0	1	2	-2
x	$\frac{1}{4}$	1	4	16	$\frac{1}{16}$

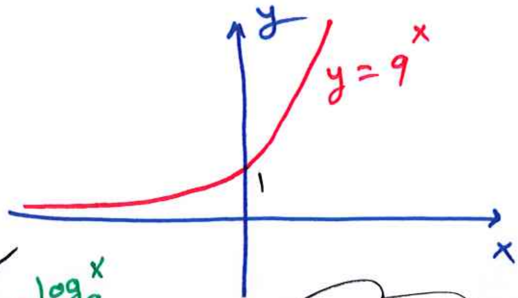


$\log_4 x$ is the inverse function of $y = 4^x$

③

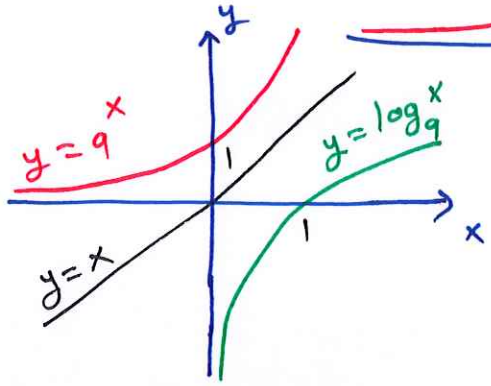


④



$y = 9^x$

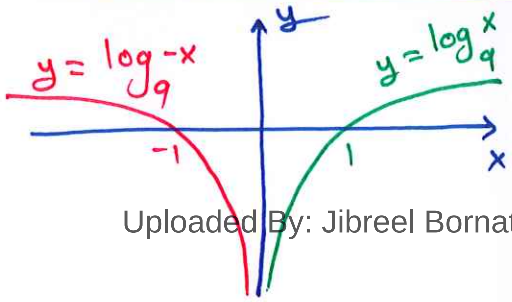
x	-1	0	1	2	-2
y	$\frac{1}{9}$	1	9	81	$\frac{1}{81}$



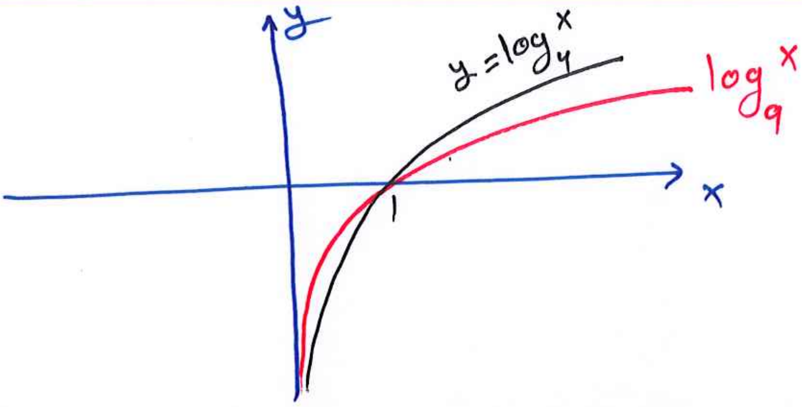
$y = 9^x$ is the inverse function of $y = \log_9 x$

⑤ $y = \log_9 (-x)$ compare with $y = \log_9 x$

$9^y = -x \Rightarrow x = -9^y$



x	$-\frac{1}{81}$	$-\frac{1}{9}$	-1	-9	-81
y	-2	-1	0	1	2



Logarithmic Properties

Assume the base $a > 0$ and $a \neq 1$. Then

1 $\log_a a^x = x$ for any real number x.

since the exponential form gives $a^x = a^x$ ✓

That is, if $a^y = a^x$ then $y = x$

Exp ① $\log_2 2^4 = 4$ ② $\ln e^3 = \log_e e^3 = 3$

③ $\log_7 7 = \log_7 7^1 = 1$ ④ $\log_9 1 = \log_9 9^0 = 0$

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2 $\log_a a = 1$ and $\log_a 1 = 0$

Exp $\log_{12} 12 = 1$ and $\log_6 1 = 0$

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3 $a^{\log_a x} = x$ for any positive real number x

Exp ① $3^{\log_3 7} = 7$ ② $e^{\ln x} = e^{\log_e x} = x$

Check $\log_3 7 = \log_3 7$ Check $\log_e x = \log_e x = \frac{\ln x}{\ln e} = \ln x$

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4 $\log_a xy = \log_a x + \log_a y$ for any positive real numbers x and y

Exp Find ① $\log_3 27 = \log_3 3^3 = 3$

② $\log_4 \frac{1}{16} = \log_4 4^{-2} = -2$

Exp Assume $\log_a x = 2$ and $\log_a y = 3$. Find $\log_a x^2 y^3$

$$\log_a x^2 y^3 = \log_a x^2 + \log_a y^3 = 2 \log_a x + 3 \log_a y$$
$$= 2(2) + 3(3)$$
$$= 4 + 9 = 13$$

5 $\log_a \frac{x}{y} = \log_a x - \log_a y$ for any $x>0$, $y>0$

Exp Find ① $\log_3 \frac{9}{27} = \log_3 9 - \log_3 27$

$= \log_3 3^2 - \log_3 3^3 = 2 - 3 = -1$

2 $\log_2 \frac{1}{\sqrt{2}} = \log_2 1 - \log_2 \sqrt{2} = 0 - \log_2 2^{\frac{1}{2}} = -\frac{1}{2}$

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6 $\log_a x^r = r \log_a x$ for any $x>0$ and $r \in \mathbb{R}$

Exp Find ① $\log_2 32 = \log_2 2^5 = 5$

② $\ln x^3 e^2 = \log_e x^3 e^2 = \log_e x^3 + \log_e e^2 = 3 \log_e x + 2$

$= 3 \frac{\ln x}{\ln e} + 2 = 3 \ln x + 2$

or
$\ln x^3 + \ln e^2$
$3 \ln x + 2 \ln e$
$3 \ln x + 2$

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7 $\log_b x = \frac{\log_a x}{\log_a b}$ convert the base b to the base a

$\log_b x = \frac{\ln x}{\ln b} = \frac{\log_e x}{\log_e b}$ convert the base b to the base e

$\log_b x = \frac{\log_{10} x}{\log_{10} b} = \frac{\log x}{\log b}$ convert the base b to the base 10

Exp Evaluate ① $\log_7 15$ using the change of base e

$\log_7 15 = \frac{\ln 15}{\ln 7} = \frac{2.71}{1.95} \approx 1.4$

② $\log_7 15$ using the change of base 10

$\log_7 15 = \frac{\log 15}{\log 7} = \frac{1.18}{0.845} \approx 1.4$