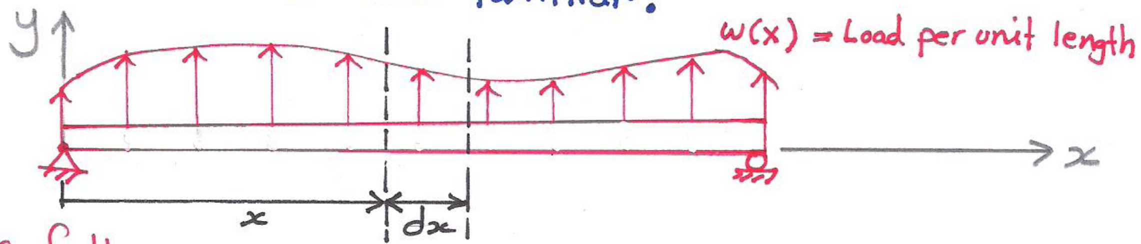
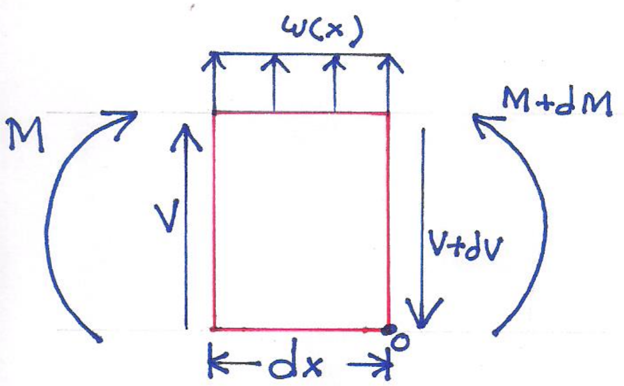
Conjugate-beam method

Conjugate-beam method is an engineering method to derive the slope and deflection of a beam. It was developed by H. Muller-Breslau in 1865. Essentially, it requires the same amount of computation as the moment-area theorems to determine a beam's slope and deflection; However the Conjugate-beam method relies only on the principles of statics, So its application will be more familiar.



Consider the following beam's infinitesimal element



$\Sigma F_y = 0 \rightarrow \boxed{\frac{dV}{dx} = w}$ (I)

$\Sigma M_o = 0 \rightarrow \boxed{\frac{dM}{dx} = V}$

From elastic beam theory we know that:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \quad \text{and} \quad \theta = \frac{dy}{dx}$$

$$\boxed{\frac{d\theta}{dx} = \frac{M}{EI}}$$

$$\boxed{\frac{dy}{dx} = \theta}$$ (II)

The basis of this method comes from the similarity of "eq.1 and eq.2" to "eq.3 and eq.4". Here the shear V compares with the slope θ, the moment M compares with the displacement y, and the external load w with the M/EI diagram.

$$\text{"Load"} \quad W \longleftrightarrow \frac{M}{EI} \quad \text{"curvature"}$$

$$\text{"Shear"} \quad V \longleftrightarrow \theta \quad \text{"slope"}$$

$$\text{"Moment"} \quad M \longleftrightarrow y \quad \text{"deflection"}$$

To make use of this comparison, we will now consider a beam having the same length as the real beam, but referred here as the "conjugate beam". The conjugate beam is loaded with $\frac{M}{EI}$ diagram derived the load on the real beam. From the above comparisons, we can state two theorems related to the conjugate beam:

Theorem I:

The slope at a point in the real beam is numerically equal to the Shear at the corresponding point in the Conjugate beam.

Theorem II:

The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam.

Conjugate - beam supports

When drawing the conjugate beam, it is important that the shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beam at its support, a consequence of theorems I and II. Corresponding real and conjugate supports are shown below

Real supports vs. conjugate supports

Real beam	Conjugate beam
 Hinged support $y=0$ $\theta\neq 0$	 Hinged support $M=0$ $V\neq 0$
 Roller support $y=0$ $\theta\neq 0$	 Roller support $M=0$ $V\neq 0$
 Fixed support $y=0$ $\theta=0$	 Free end $M=0$ $V=0$
 Free end $y\neq 0$ $\theta\neq 0$	 Fixed support $M\neq 0$ $V\neq 0$
 Middle support $y=0$ $\theta\neq 0$	 Middle hinge $M=0$ $V\neq 0$

Real beam	Conjugate beam
 Middle hinge $y\neq 0$ $\theta\neq 0$ "discontinue" $\theta_L\neq\theta_R$	 Middle support $M\neq 0$ $V\neq 0$ "discontinue" $V_L\neq V_R$
 $y=0$ $\theta_L\neq\theta_R$	 $M=0$ $V_L\neq V_R$
 $y\neq 0$ $\theta=0$	 $M\neq 0$ $V=0$

Examples on conjugate beams

	Real beam	Conjugate beam
Simple beam		
Cantilever beam		
Left-end overhanging beam		
Both-end over hanging beam		
Gerber's beam (2 span)		
Gerber's beam (3 span)		

* If a beam is determinate, its conjugate beam will be determinate.

* If a beam is indeterminate, its conjugate beam will be unstable

The procedure for conjugate-beam method

a) Draw the bending moment diagram for the real beam, and then the $\frac{M}{EI}$ diagram.

b) Draw the conjugate beam. This beam has the same length as the real beam and has the corresponding supports shown above.

c) Apply a load of M/EI on the conjugate beam. This loading is assumed to be distributed over the conjugate beam and is directed upward when M/EI is positive and downward when M/EI is negative. In other words, the loading always acts away from the beam.

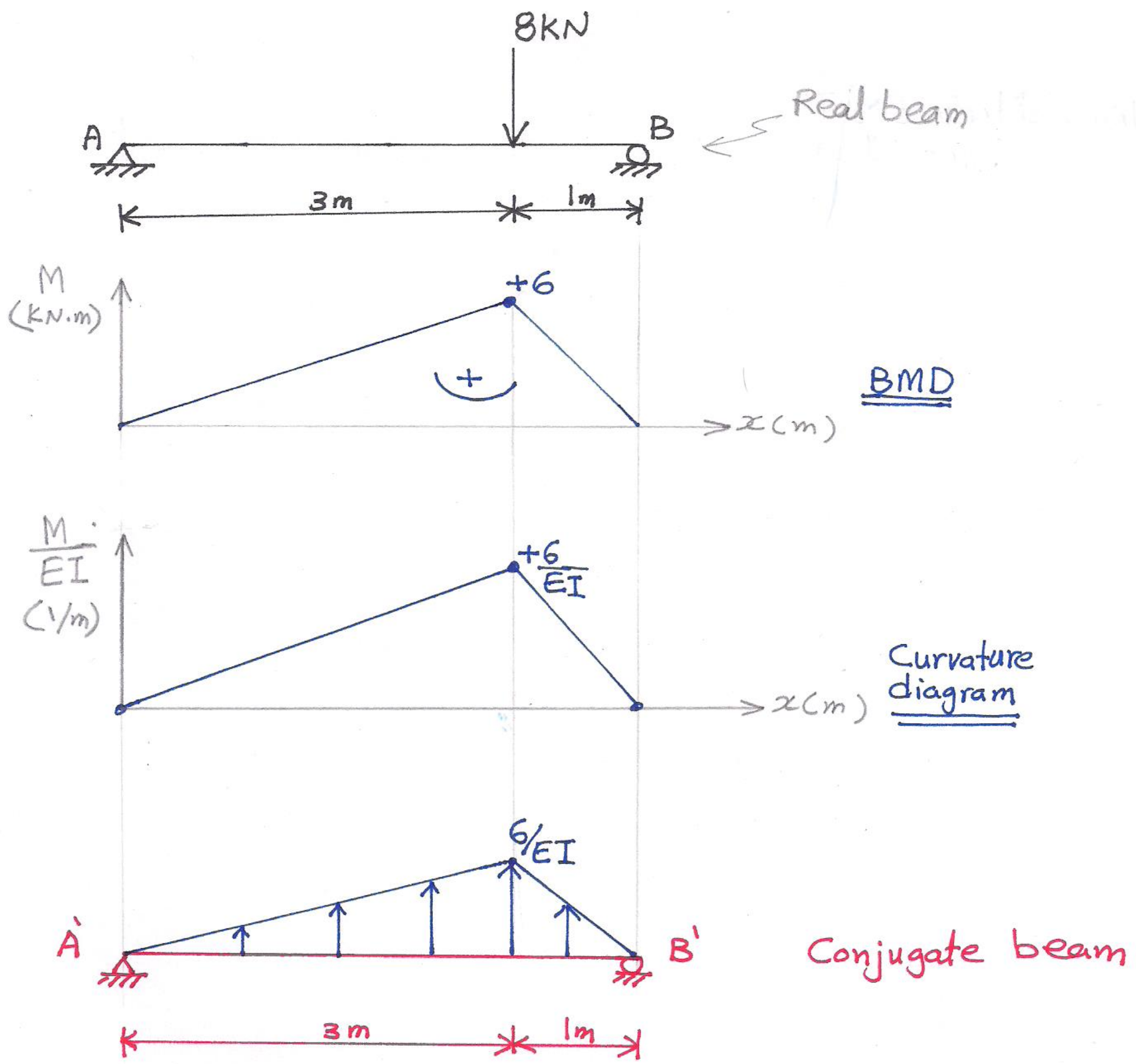
d) Using the equations of statics, determine the reactions at the conjugate beams supports.

e) Section the conjugate beam at the point where the slope θ and displacement y of the real beam are to be determined. At the section, show the unknown shear V' and M' equal to θ and y, respectively, for the real beam.

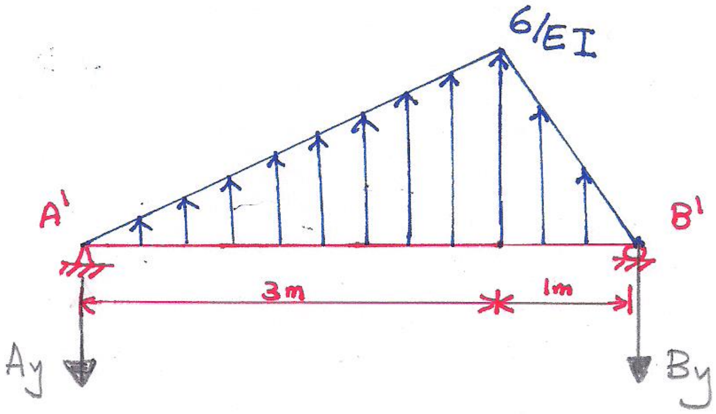
In particular, if these values (θ, y) are positive, the slope is counterclockwise and the displacement is upward.

Example

For the beam shown, determine (a) slope at A and B, (b) The maximum deflection. EI = constant.



Determine the reactions at points A' and B' on the Conjugate beam.



$$\circlearrowleft^{+} \Sigma M_{A'} = 0 : -4\,B_y + \left(\tfrac{1}{2}\right)(3)\left(\tfrac{6}{EI}\right)\left(\tfrac{2}{3}*3\right) + \left(\tfrac{1}{2}\right)(1)\left(\tfrac{6}{EI}\right)\left(3+\tfrac{1}{3}*1\right) = 0$$

$$B_y = +\frac{7}{EI}$$

$$+\uparrow \Sigma F_y = 0 : -A_y - \frac{7}{EI} + \left(\tfrac{1}{2}\right)(4)\left(\tfrac{6}{EI}\right) = 0$$

$$A_y = +\frac{5}{EI}$$

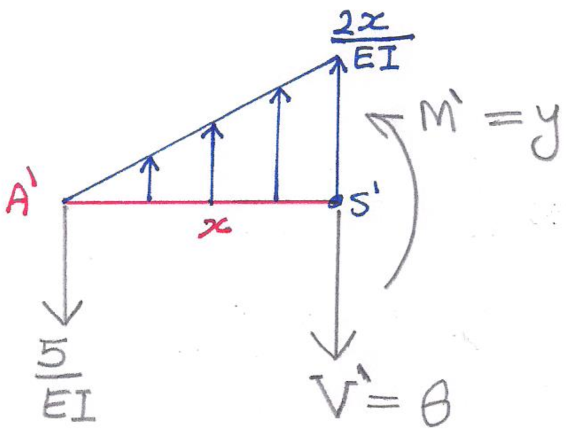
$\Rightarrow$ Slope at point A on real beam = shear at A' on conjugate beam

$$\theta_A = V_{A'} = \frac{-5}{EI} \text{ rad}$$

$\Rightarrow$ Slope at point B on real beam = shear at B' on conjugate beam

$$\theta_B = V_{B'} = \frac{+7}{EI} \text{ rad}$$

For the maximum deflection, make a section cut at distance x measured from point A' on conjugate beam to locate the point at which shear $= 0$.



$$+\uparrow \sum F_y = 0: \quad \frac{-5}{EI} + \frac{1}{2}(x)\left(\frac{2x}{EI}\right) - \theta = 0 \Rightarrow \theta = \frac{x^2}{EI} - \frac{5}{EI}$$

$\theta = 0$ (for maximum deflection) $\rightarrow x^2 - 5 = 0$

$$x = \sqrt{5}\ m < 3\ m$$

$$\sum M_{s'} = 0: \quad +y - \left(\frac{1}{2}\right)(x)\left(\frac{2x}{EI}\right)\left(\frac{x}{3}\right) + \left(\frac{5}{EI}\right)(x) = 0$$

$$y = \frac{x^3}{3EI} - \frac{5x}{EI}$$

This is the equation of the elastic curve for $0 \le x \le 3m$

at $x = \sqrt{5}\ m \rightarrow y = \frac{(\sqrt{5})^3}{3\,EI} - \frac{5\sqrt{5}}{EI} = -\frac{7.454}{EI}\ (\downarrow)$