

7.4 Exponential Change

P.E

I.V.P :- $\frac{dy}{dt} = ky$

$$\frac{dy}{y} = k dt$$

$$\ln|y| = kt + c$$

$$\rightarrow y = y_0 e^{kt}$$

$$\rightarrow y = y_0 e^{-kt}$$

I.C

$y = y_0 = y(t_0), t_0 = 0$

Exponential growth, $k > 0$

Exponential decay, $k < 0$

Q28 Voltage in a discharging capacitor. Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage across its terminals and that if t is measured in seconds

$$\frac{dV}{dt} = -\frac{1}{40} V$$

Sol: Find V ? How long will it take the voltage to drop to 10% of the original value?

$$V = V_0 e^{-\frac{1}{40}t}$$

$$10\% V_0 = V_0 e^{(-1/40)t}$$

$$0.10 = e^{(-1/40)t}$$

$$\rightarrow t = 40 \ln 10 = \ln 10^{40} \text{ sec.}$$

30 Growth of Bacteria

At the end of 3 hours there are 10000 bacteria

At the end of 5 hours there are 40000

How many bacteria were present initially?

Sol: $y = y_0 e^{kt}$ with $y(3) = 10000$ $y(5) = 40000$

$$\begin{aligned} \rightarrow 10000 &= y_0 e^{3k} \\ \rightarrow 40000 &= y_0 e^{5k} \end{aligned} \Rightarrow 4y_0 e^{3k} = y_0 e^{5k}$$

$$\Rightarrow e^{2k} = 4$$

$$\Rightarrow 2k = \ln 4 \Rightarrow \boxed{k = \ln 2}$$

$$y = y_0 e^{(\ln 2)t}$$

Now, use $y(3) = 10000$ to find y_0

$$10000 = y_0 e^{3 \ln 2}$$

$$y_0 = \frac{10000}{8} = 1250 \text{ bacteria}$$

[31] The incidence of a disease :- (Continuation of Exp 3)

Suppose that in any given the number of cases can be reduced by 25% instead of 20%.

① How long will it take to reduce the number of cases to 1000??

We assume $y = y_0 e^{kt}$

If we count from today, then $y_0 = 10000$ at $t=0$

$$y = 10000 e^{kt}$$

$$7500 = 10000 e^k$$

$$0.75 = e^k \rightarrow \boxed{k = \ln 0.75} \quad \begin{matrix} k < 1 \\ \text{exp. decay} \end{matrix}$$

Now, $y = y_0 e^{\ln 0.75 t}$ ----- solve it for t .

$$1000 = 10000 e^{\ln 0.75 t}$$

$$\ln 0.1 = \ln 0.75 t$$

$$\boxed{t = 8 \text{ years}}$$

② How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1??

$$y = 10000 e^{\ln 0.75 t}$$

$$1 = 10000 e^{\ln 0.75 t}$$

$$t = 32.02 \text{ years}$$

more than 32 years
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[36] The half life of Polonium is 139 days

but your sample will not be useful to you after 95% of the sample has disintegrated. For about How many days after the sample arrives will you be able to use the Polonium??

Sol: $y = y_0 e^{-kt}$

$$T = \text{half life} = 139 \text{ day} \implies k = \frac{\ln 2}{T} = \frac{\ln 2}{139}$$
$$\implies k = 0.00499$$

$$y = y_0 e^{-0.00499t}$$

$$0.05 y_0 = y_0 e^{-0.00499t}$$

$$t = \frac{\ln 0.05}{-0.00499} = 600.34$$

$$t \approx 600 \text{ days}$$

7.5 Indeterminate forms and L'Hopital Rule:-

10 Use L'Hopital Rule to evaluate the limit:-

$$\boxed{10} \quad \lim_{t \rightarrow 1} \frac{t^3 - 1}{4t^3 - t - 3} = \frac{0}{0}$$

$$\lim_{t \rightarrow 1} \frac{3t^2}{12t^2 - 1} = \frac{3}{11}$$

$$\boxed{21} \quad \lim_{x \rightarrow 0} \frac{x^2}{\ln(\sec x)} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2x}{\frac{1}{\sec x} \sec x \tan x} &= \lim_{x \rightarrow 0} \frac{2x}{\tan x} = \lim_{x \rightarrow 0} \frac{2}{\sec^2 x} \\ &= \frac{2}{(1)^2} = 2 \end{aligned}$$

$$\boxed{27} \quad \lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} - 1}{\theta} = \frac{0}{0}$$

$$= \lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} (\ln 3) (\cos \theta)}{1}$$

$$= (\ln 3) \left(3^0 \right) (1) = \ln 3$$

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$$\lim_{x \rightarrow 0} \frac{3^x - 1}{2^x - 1} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{3^x \ln 3}{2^x \ln 2} = \frac{\ln 3}{\ln 2} = \boxed{\log_2 3}$$

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$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln x} = \frac{-\infty}{-\infty}$$

$$\lim_{x \rightarrow 0^+} \frac{e^x}{e^x - 1} \bigg/ \frac{1}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{x e^x}{e^x - 1}$$

$$= \lim_{x \rightarrow 0^+} \frac{x e^x + e^x}{e^x} = \lim_{x \rightarrow 0^+} x + 1 = \boxed{1}$$

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$$\lim_{x \rightarrow 0^+} (\ln x - \ln \sin x) = -\infty + \infty = \infty - \infty$$

$$= \lim_{x \rightarrow 0^+} \ln \left(\frac{x}{\sin x} \right)$$

$$= \ln \left[\lim_{x \rightarrow 0^+} \frac{x}{\sin x} \right] = \frac{0}{0}$$

$$= \ln \left[\lim_{x \rightarrow 0^+} \frac{1}{\cos x} \right] = \ln(1) = \boxed{0}$$

Another solution:-

$$\lim_{x \rightarrow 0^+} \frac{x/x}{\sin x/x}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{\frac{\sin x}{x}}$$

$$= \frac{1}{1} = \boxed{1}$$

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$$\lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) \quad \frac{1}{\text{small}^+} - \frac{1}{\text{small}^+}$$

$\infty - \infty$

$$= \lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{\ln x (x-1)} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{\ln x (1) + (x-1) \cdot \frac{1}{x}}$$

$$= \lim_{x \rightarrow 1^+} \frac{1-x}{x \ln x + x - 1} \quad = \frac{0}{0}$$

$$= \lim_{x \rightarrow 1^+} \frac{-1}{\frac{x}{x} + \ln x + 1}$$

$$= \lim_{x \rightarrow 1^+} \frac{-1}{2 + \ln x} = \frac{-1}{2+0} = \boxed{-\frac{1}{2}}$$

42 $\lim_{x \rightarrow 0^+} (\csc x - \cot x + \cos x) \quad \frac{1}{\text{small}^+} - \frac{1}{\text{small}^+} + 1$

$\infty - \infty + 1$

$$= \lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) + \lim_{x \rightarrow 0^+} \cos x$$

$$= \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{\sin x} \quad \frac{0}{0}$$

$$+ \lim_{x \rightarrow 0^+} \cos x$$

$$= \lim_{x \rightarrow 0^+} \frac{+\sin x}{\cos x}$$

$$+ \lim_{x \rightarrow 0^+} \cos x$$

$$= 0 + 1 = \boxed{1}$$

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$$\lim_{x \rightarrow \infty} x^2 e^{-x}$$

$\infty \cdot 0$

$$= \lim_{x \rightarrow \infty} \frac{x^2}{e^x}$$

$$= \lim_{x \rightarrow \infty} \frac{2x}{e^x}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{e^x} = \frac{2}{\infty} = \boxed{0}$$

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$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \tan x} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sec^2 x + \tan x} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{2x \cdot \sec^2 x \tan x + \sec^2 x + \sec^2 x}$$

$$= \frac{0}{0+2} = \boxed{0}$$

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$$\lim_{x \rightarrow 1^+} x^{1/x-1}$$

$$(1)^\infty$$

$$* f(x) = x^{\frac{1}{x-1}}$$

$$\rightarrow \ln f(x) = \frac{1}{x-1} \ln x$$

$$\rightarrow \lim_{x \rightarrow 1^+} \ln f(x) = \lim_{x \rightarrow 1^+} \frac{\ln x}{x-1} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 1^+} \frac{\frac{1}{x}}{1}$$

$$= 1$$

$$\therefore \lim_{x \rightarrow 1^+} x^{\frac{1}{x-1}} = e^1 = \boxed{e}$$

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$$\lim_{x \rightarrow \infty} (\ln x)^{1/x}$$

$$* f(x) = (\ln x)^{\frac{1}{x}}$$

$$* \ln f(x) = \frac{1}{x} \ln(\ln x)$$

$$* \lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = 0$$

$$\therefore \lim_{x \rightarrow \infty} (\ln x)^{1/x} = e^0 = 1$$

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$$\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x = (\infty^0)$$

$$* f(x) = \left(1 + \frac{1}{x}\right)^x$$

$$* \ln f(x) = x \ln \left(1 + \frac{1}{x}\right)$$

$$* \lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{1 + 1/x} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x+1}$$

$$= 0$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x = e^0 = 1$$

Theorem $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

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$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 2} \right)^{1/x} \quad (\infty^0)$$

$$* f(x) = \left(\frac{x^2 + 1}{x + 2} \right)^{1/x}$$

$$* \ln f(x) = \frac{1}{x} \ln \left(\frac{x^2 + 1}{x + 2} \right)$$

$$* \lim_{x \rightarrow \infty} (\ln f(x)) = \lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x^2 + 1}{x + 2} \right)}{x}$$

$$= \lim_{x \rightarrow \infty} \frac{\ln(x^2 + 1) - \ln(x + 2)}{x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2x}{x^2 + 1} - \frac{1}{x + 2}}{1}$$

$$= \lim_{x \rightarrow \infty} \frac{2x(x + 2) - (x^2 + 1)}{(x^2 + 1)(x + 2)} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{2x^2 + 4x - x^2 - 1}{x^3 + 2x^2 + x + 2}$$

$$= \lim_{x \rightarrow \infty} \frac{4x + 4 - 2x}{3x^2 + 4x + 1} = \lim_{x \rightarrow \infty} \frac{2x + 4}{3x^2 + 4x + 1}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{6x + 4} = 0$$

$\therefore \lim_{x \rightarrow \infty} f(x) = e^0 = 1$

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$$\lim_{x \rightarrow 0^+} (\sin x) (\ln x)$$

$0 \cdot \infty$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} \quad \frac{-\infty}{\infty}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}$$

$$= \lim_{x \rightarrow 0^+} \frac{1/x}{-\frac{1}{\sin x} \cdot \frac{1}{\tan x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x \tan x}{x} \quad \frac{0}{0}$$

$$= - \lim_{x \rightarrow 0^+} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0^+} \tan x$$

$$= -1 \cdot 0 = \boxed{0}$$

Th.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

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$$\lim_{x \rightarrow -\infty} \frac{2^x + 4^x}{5^x - 2^x} \quad \frac{0}{0} \text{ l'Hopital Rule}$$

$$= \lim_{x \rightarrow -\infty} \frac{2^x \left[1 + \left(\frac{4}{2}\right)^x \right]}{2^x \left[\left(\frac{5}{2}\right)^x - 1 \right]}$$

$$= \lim_{x \rightarrow -\infty} \frac{1 + \left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x - 1}$$

$$= \lim_{x \rightarrow -\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \frac{1+0}{0-1} = \boxed{-1}$$

Another solution:-

$$\lim_{x \rightarrow -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim_{x \rightarrow -\infty} \frac{2^x \ln 2 + 4^x \ln 4}{5^x \ln 5 - 2^x \ln 2}$$

$$= \lim_{x \rightarrow -\infty} \frac{\ln 2 + 2^x \ln 4}{\left(\frac{5}{2}\right)^x \ln 5 - \ln 2}$$

$$= \frac{\ln 2}{-\ln 2} = -1$$