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MATH1321

Calculus 2

Chapter 10.8

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10.8 Taylor and Maclaurin Series

Let f be a smooth function (all derivatives exist) on an interval that contains the interior point a. Then

1) The Taylor series generated by f has $x = a$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

2) The Maclaurin series generated by f is: (where $x = 0$)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Note that the maclaurin series genarated by f is the Taylor series genarated by f at $x = 0$.

* The function $f(x)$ could be approximated by the Taylor Polynomials: $P_0(x), P_1(x), \dots, P_n(x)$.

$P_0(x) = f(a)$. Polinomial of degree 0.

$P_1(x) = f(a) + f'(a)(x-a)$. Polinomial of degree 1. (Linearization)

$P_2(x) = P_1(x) + \frac{f''(a)}{2!}(x-a)^2$. Polinomial of degree 2.

$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$. Polinomial of degree 3.

$\vdots$

$P_n(x) = P_0(x) + P_1(x) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ Polinomial of degree n.

Ex. Find Maclaurin series for? (MS = TS at $x=0$).

1) $f(x) = \cos x$ we need $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \underset{1}{f(0)} + \underset{0}{f'(0)}x + \frac{\overset{-1}{f''(0)}}{2!}x^2 + \dots$

$f'(x) = -\sin x$ $f'(0) = 0$ $f''(x) = \cos x$ $f''(0) = 1$

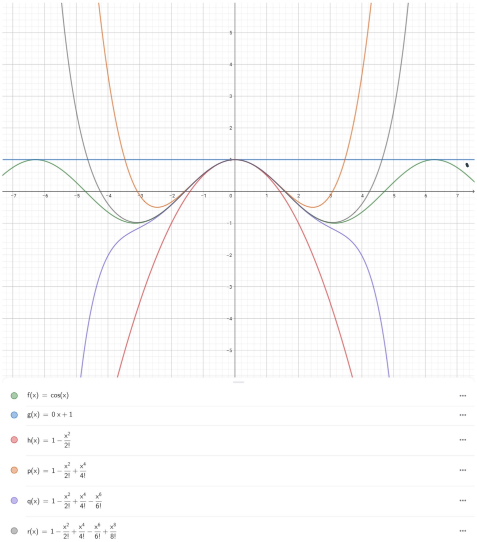
$f'''(x) = \sin x$ $f'''(0) = 0$ $f''(x) = -\cos x$ $f''(0) = -1$

$f^{(5)} = -\sin x$ $f^{(5)}(0) = 0$ $f^{(4)}(x) = \cos x$ $f^{(4)}(0) = 1$

$\vdots$ $\vdots$

Therefore $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \dots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$



Approximation:

$P_0(x) = 1$. Poly. of degree 0

$P_2(x) = 1 - \frac{x^2}{2!}$ Poly. of degree 2

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ Poly. of degree 4

The approximation is shown in the figure $\longrightarrow$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ #

$P_n(x) = 1 - \frac{x^2}{2!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$ Poly of degree n.

$\lim_{n \to \infty} P_n(x) = \cos x$ about 0.

2) $\sin x$ we need to find $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \underset{0}{f(0)}+\underset{1}{f'(0)}x+\underset{0}{\frac{f''(0)}{2!}}x^2+\dots$

$f(x)=\sin x$ $f(0)=0$ $\quad$ $f'(x)=\cos x$ $f'(0)=1$

$f''(x)=-\sin x$ $f''(0)=0$ $\quad$ $f'''(x)=-\cos x$ $f'''(0)=-1$

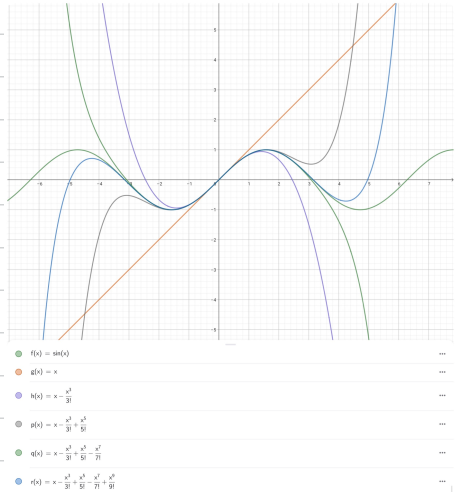
$f^{(4)}(x)=\sin x$ $f^{(4)}(0)=0$ $\quad$ $f^{(5)}(x)=\cos x$ $f^{(5)}(0)=1$

$\vdots$

Therefore $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = 0+x+0-\frac{x^3}{3!}+0+\frac{x^5}{5!}+\dots$

$= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\dots$

$\therefore \sin x = x-\frac{x^3}{3!}+\dots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ #



3) $f(x)=e^x$ we need to find $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \underset{1}{f(0)}+\underset{1}{f'(0)}x+\underset{1}{\frac{f''(0)}{2!}}x^2+\dots$

$f(x)=e^x$ $f(0)=1$ $\quad$ $f'(x)=e^x$ $f'(0)=1$

$f''(x)=e^x$ $f''(0)=1$ $\quad$ $f'''(x)=e^x$ $f'''(0)=1$

$\vdots$

Therefore $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$

$\therefore e^x = 1+x+x^2+x^3+\dots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ #

Maclaurin Series of e^x, $\cos x$ and $\sin x$:

1) $e^x = 1+x+\frac{x^2}{2!}+\dots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$.

2) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

3) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Ex. Find Taylor Series of $f(x) = 2^x$ at $x = 1$? ($x = 1$ means $a = 1$)

*** Reminder ***

$f(x) = 2^x$, $f'(x) =$ (زي ما هو x مشتقة اي فوق x ln اي تحتها)

we need to find $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots$

$f(x) = 2^x$ $\quad f'(x) = 2^x \ln 2$

$f''(x) = 2^x(\ln 2)^2$ $\quad f'''(x) = 2^x(\ln 2)^3$ so $f^{(n)}(x) = 2^x(\ln 2)^n$ $\quad f(1) = 2 \,/\, f'(1) = 2(\ln 2) \dots$

Therefore $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = 2 + 2(\ln 2)(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \dots$

so $2 + 2(\ln 2)(x-1) + \dots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(x-1)^n$ #

Ex. Find MS for $\cosh x$?

$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + e^{-x})$

$= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots\right)\right)$

$= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)$

$= \frac{1}{2}\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \dots\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

we can find it using the previous method.

Ex. Find the first 4 nonzero terms in the MS of ?

1) $\frac{1}{3}(2x + x\cos x) = \frac{2}{3}x + \frac{x}{3}\cos x$

$= \frac{2}{3}x + \frac{x}{3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$

$= \frac{2}{3}x + \frac{x}{3} - \frac{x^3}{(3)2!} + \frac{x^5}{(3)5!} - \frac{x^7}{(3)6!} + \dots$

$= x - \frac{x^3}{(3)2!} + \frac{x^5}{(3)4!} - \frac{x^7}{(3)6!} + \dots$

The first 4 nonzero terms: $\left\{x, \frac{-x^3}{(3)2!}, \frac{x^5}{(3)4!}, \frac{x^7}{(3)6!}\right\}$.

2) $e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$

we need only the first 4 nonzero terms.

$= x + x^2 + \frac{x^3}{2!} + \cancel{\frac{x^4}{3!}} + \dots - \frac{x^3}{3!} - \cancel{\frac{x^4}{3!}} - \frac{x^5}{3!\,2!} - \frac{x^6}{3!\,3!} + \dots + \frac{x^5}{5!} + \frac{x^6}{5!} + \dots$

$= x + x^2 + \left(\frac{x^3}{2!} - \frac{x^3}{3!}\right) + \left(\frac{x^5}{4!} - \frac{x^5}{3!\,2!} + \frac{x^5}{5!}\right) + \dots$

The first 4 nonzero terms: $\left\{x, x^2, \frac{x^3}{2!} - \frac{x^3}{3!}, \frac{x^5}{4!} - \frac{x^5}{3!\,2!} + \frac{x^5}{5!}\right\}$

3) $\cos 2x$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \dots$

The first 4 nonzero terms: $\left\{1, \frac{4x^2}{2!}, \frac{16x^2}{4!}, \frac{(2x)^6}{6!}\right\}$.