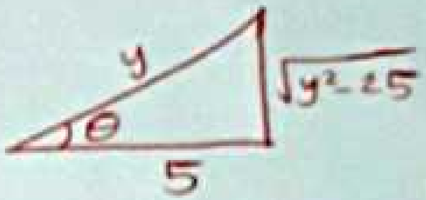
Sec 8.3 : Trigonometric Substitutions

$\boxed{12}$ $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy$

$\sec\theta = \frac{y}{5}$



let $y = 5\sec\theta$

$dy = 5\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sqrt{25\sec^2\theta - 25}}{125\sec^3\theta}\cdot 5\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{5\sqrt{\sec^2\theta - 1}}{125\sec^3\theta}\cdot 5\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{25\tan\theta\cdot\sec\theta\tan\theta}{125\sec^3\theta}\,d\theta$$

$$= \int \frac{\tan^2\theta}{5\sec^2\theta}\,d\theta = \frac{1}{5}\int \frac{\sin^2\theta/\cos^2\theta}{1/\cos^2\theta}\,d\theta$$

$$= \frac{1}{5}\int \underbrace{\sin^2\theta}_{\downarrow}\,d\theta$$

$$= \frac{1}{5}\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{10}\left[\theta - \underbrace{\frac{\sin(2\theta)}{2}}_{\frac{2\sin\theta\cos\theta}{2}}\right] + C$$

$$= \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{y}\cdot\frac{5}{y}\right] + C$$

$\boxed{18}$ $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}$$

$$= \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta \, \sec\theta} = \int \frac{\sec\theta}{\tan^2\theta} \, d\theta$$

$$= \int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta} \, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

let $u = \sin\theta$

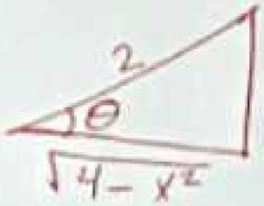
$du = \cos\theta \, d\theta$

$$= \int \frac{du}{u^2} = \int u^{-2} du = \frac{-1}{u} + C$$

$$= \frac{-1}{\sin\theta} + C$$ من المثلث

$$= -\frac{\sqrt{x^2+1}}{x} + C$$

$\boxed{24}$ $\displaystyle\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$



$\sin\theta = \frac{x}{2}$

let $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$= \displaystyle\int \frac{2\cos\theta}{(4-4\sin^2\theta)^{3/2}} d\theta$

$= \displaystyle\int \frac{2\cos\theta}{(4\cos^2\theta)^{3/2}} d\theta$

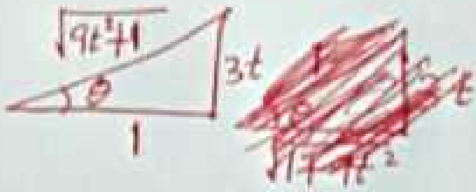
$= \displaystyle\int \frac{2\cos\theta}{8\cos^3\theta} d\theta = \frac{1}{4}\int \sec^2\theta \, d\theta$

$= \frac{1}{4}\tan\theta$

$= \frac{1}{4} \cdot \frac{x}{\sqrt{4-x^2}} \Big|_0^1$

$= \frac{1}{4}\left[\frac{1}{\sqrt{3}} - 0\right] = \frac{4}{\sqrt{3}}$

$\boxed{30}$ $\displaystyle\int \frac{6}{(9t^2+1)^2} dt$



$\tan\theta = 3t$

$\to \theta = \tan^{-1}(3t)$

let $3t = \tan\theta$

$3dt = \sec^2\theta \, d\theta$

$= \displaystyle\int \frac{6}{(\tan^2\theta+1)^2} \cdot \frac{\sec^2\theta}{3} d\theta = 2\int \frac{\sec^2\theta}{(\sec^2\theta)^2} d\theta$

$= 2\displaystyle\int \frac{1}{\sec^2\theta} d\theta$

$$= 2\int \cos^2\theta \, d\theta$$

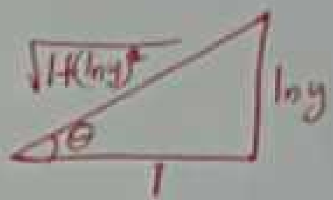
$$= \cancel{2}\int \frac{1+\cos 2\theta}{\cancel{2}}\, d\theta$$

$$= \theta + \frac{\sin(2\theta)}{2} + C_1 \longrightarrow \frac{2\sin\theta\cos\theta}{2} = \sin\theta\cos\theta$$

$$= \tan^{-1}(3t) + \frac{3t}{\sqrt{9t^2+1}} \cdot \frac{1}{\sqrt{9t^2+1}} + C_1$$

$$= \tan^{-1}(3t) + \frac{3t}{9t^2+1} + C_1 .$$

$\boxed{38}$ $$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$$



let $\ln y = \tan\theta$

$$\frac{1}{y}\cdot dy = \sec^2\theta \, d\theta \longrightarrow dy = y\sec^2\theta \, d\theta$$

$$= \int \frac{\cancel{y}\sec^2\theta \, d\theta}{\cancel{y}\sqrt{1+(\tan\theta)^2}} = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta|$$

$$= \ln\left|\sqrt{1+(\ln y)^2} + \ln y\right| \Big|_1^e$$

$$= \ln\left(\sqrt{1+(\ln e)^2} + \ln e\right) - \ln\left(\sqrt{1+(\ln 1)^2} + \ln 1\right)$$

$$= \ln(\sqrt{2}+1) - \ln 1 = \ln(\sqrt{2}+1) .$$

- Sec 8.4 : Integration Of Rational Functions By Partial Fractions.

$\boxed{13}$ $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$

$$= \int \frac{2x+1}{(x-4)(x-3)} = \int \frac{A}{x-4} + \frac{B}{x-3}\,dx$$

$\rightarrow \quad A(x-3)+B(x-4) = 2x+1$

• $\boxed{A=9}$, $\boxed{B=-7}$

$$= \int \frac{9}{x-4}\,dx + \int \frac{-7}{x-3}\,dx$$

$$= 9\ln|x-4| - 7\ln|x-3| + C$$

[illegible]

$$= \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C.$$

$\boxed{20}$ $\displaystyle\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx$

$$= \int \frac{x^2}{(x-1)(x+1)^2}\,dx$$

$$= \int \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\, dx$$

[illegible]

$$\rightarrow A(x+1)^2 + B(x-1)(x+1) + C(x-1) = x^2$$

$$Ax^2 + 2Ax + A + Bx^2 - B + Cx - C = x^2$$

$$A + B = 1 \longrightarrow \boxed{B = 1 - A}$$

$$2A + C = 0 \longrightarrow \boxed{C = -2A}$$

$$A - B - C = 0$$

$$\hookrightarrow A - (1 - A) - (-2A) = 0$$

$$4A - 1 = 0 \longrightarrow \boxed{A = \frac{1}{4}}$$

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$$\boxed{B = \frac{3}{4}} \quad , \quad \boxed{C = \frac{-2}{4} = \frac{-1}{2}}$$

يستطيع إيجاد A,B,C
by cover method

$$\int \frac{x^2}{(x-1)(x^2+2x+1)}\, dx = \int \left[\frac{1/4}{x-1} + \frac{3/4}{x+1} + \frac{-1/2}{(x+1)^2}\right] dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| - \frac{1}{2}\,\frac{(x+1)^{-1}}{-1} + C$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$$

$\boxed{23}$

$$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$$

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$$= \int \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2}\,dy$$

$$\longrightarrow (Ay+B)(y^2+1) + Cy + D = y^2+2y+1$$

$$Ay^3 + Ay + By^2 + B + Cy + D = y^2+2y+1$$

$$\boxed{A=0}$$

$$\boxed{B=1}$$

$$A+C=2 \longrightarrow 0+C=2 \longrightarrow \boxed{C=2}$$

$$B+D=1 \longrightarrow 1+D=1 \longrightarrow \boxed{D=0}$$

$$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1} + \frac{2y}{(y^2+1)^2}\,dy$$

$$\int \frac{2y}{(y^2+1)^2}\,dy = \quad \text{let } u = y^2+1$$

$$= \int \frac{du}{u^2} = \frac{-1}{u} = \frac{-1}{y^2+1}$$

$$= \tan^{-1} y + \frac{-1}{y^2+1} + C.$$

$\boxed{37}$ $\displaystyle\int \frac{y^4+y^2-1}{y^3+y}\,dy$

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$$\begin{array}{r|l} & y \\ \hline y^3+y & \cancel{y^4}+\cancel{y^2}-1 \\ & \underline{-\cancel{y^4} \mp \cancel{y^2}} \\ & -1 \end{array}$$

$$= y + \frac{-1}{y^3+y}$$

$$\therefore \int \frac{y^4+y^2-1}{y^3+y}\,dy = \int y + \frac{-1}{y^3+y}\,dy$$

$$= \int y + \frac{-1}{y(y^2+1)}\,dy$$

by partial fraction.

$$\frac{-1}{y^3+y} = \frac{A}{y} + \frac{By+C}{y^2+1}$$

$$\longrightarrow Ay^2 + A + By^2 + Cy = -1$$

$A+B=0 \longrightarrow \boxed{A=-B}$

$\boxed{C=0}$

$\boxed{A=-1}$

$\therefore \boxed{B=1}$

$$\int \frac{y^4+y^2-1}{y^2+y}\,dy = \int y + \frac{-1}{y} + \frac{y}{y^2+1}\,dy$$

$\frac{1}{2}\int \frac{2y}{y^2+1}\,dy = \frac{1}{2}\ln(y^2$

$$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(y^2+1) + C$$

$\boxed{47}$ $\int \frac{\sqrt{x+1}}{x}\,dx$

let $\sqrt{x+1} = u \longrightarrow x = u^2-1$

$\frac{dx}{2\sqrt{x+1}} = du \longrightarrow dx = 2\sqrt{x+1}\,du$

$= 2u\,du$

$$= \int \frac{u}{u^2-1}\cdot 2u\,du = \int \frac{2u^2}{u^2-1}\,du$$

$$\begin{array}{r|l} & 2 \\ \hline u^2-1 & 2u^2 \\ & -2u^2 \pm 2 \\ \hline & 2 \end{array}$$

$$= \int 2 + \frac{2}{u^2-1}\,du$$

partial fraction

$\int \frac{2\,\text{[redacted]}}{u^2-1}\,du = \int \frac{A}{u-1} + \frac{B}{u+1}\,du$

$A(u+1) + B(u-1) = 2$

$\longrightarrow A = 1 \quad , \quad B = -1$

$$= \int 2 + \frac{1}{u-1} + \frac{-1}{u+1}\, du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2\sqrt{x+1} + \ln|\sqrt{x+1} - 1| - \ln|\sqrt{x+1} + 1| + C.$$

$$= \boxed{2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1}\right| + C}.$$