Note Title ٢٣/٠٥/٢١

Review of Ch 10

Answer the following questions

1) Find the following limits:

a) $\lim \dfrac{n^2 \ln n}{3^n} \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim \dfrac{n^2 \cdot \frac{1}{n} + 2n \ln n}{3^n \ln 3}$

$= \lim \dfrac{n + 2n\ln n}{3^n \ln 3} \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim\limits_{n\to\infty} \dfrac{1 + 2n\cdot\frac{1}{n} + 2\ln n}{3^n \ln^2 3}$

$= \lim\limits_{n\to\infty} \dfrac{3 + 2\ln n}{3^n \ln^2 3} \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim \dfrac{2/n}{3^n \ln^3 3}$

$= \lim\limits_{n\to\infty} \dfrac{2}{n\, 3^n \ln^3 3} = 0$

حل 2

ممكن استخدام نظرية Sandwich إذا لاحظنا ان

$1 < \ln n < n^r$ لأن $r > 0$ وذلك عندما n تكون كبيرة بشكل كافٍ

بأخذ $r = 1$ فإن

$$\frac{n^2}{3^n} < \frac{n^2 \ln n}{3^n} < \frac{n^3}{3^n}$$

as $n \to \infty$ each term $\to 0$

by L.R. $\lim\limits_{n\to\infty} \dfrac{n^2}{3^n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{n^3}{3^n} = 0$.

so by Sandwich Thrm, $\lim \dfrac{n^2 \ln n}{3^n} = 0$.

b) $\lim \left(3^n + 5^n\right)^{\frac{1}{n}} \quad (\infty^0)$

consider $\lim \ln \left(3^n + 5^n\right)^{\frac{1}{n}} = \lim \dfrac{\ln(3^n + 5^n)}{n} \left(\frac{\infty}{\infty}\right)$

$\overset{L.R.}{=} \lim \left(\dfrac{3^n \ln 3 + 5^n \ln 5}{3^n + 5^n}\right) = \lim \dfrac{\left(\frac{3}{5}\right)^n \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^n + 1} = \boxed{\ln 5}$

c) $\lim \dfrac{3^n\, 6^n}{2^{-n}\, n!} = \lim \dfrac{2^n\, 3^n\, 6^n}{n!} = \lim \dfrac{(36)^n}{n!} = 0$

2) Find the sum of the following Series:

a) $\sum_{n=0}^{\infty} e^{-2n}(2^n - 1) = \sum_{n=0}^{\infty}\left(\dfrac{2^n}{e^{2n}} - \dfrac{1}{e^{2n}}\right)$

$= \sum_{n=0}^{\infty}\left(\left(\dfrac{2}{e^2}\right)^n - \left(\dfrac{1}{e^2}\right)^n\right)$

$\sum_{n=0}^{\infty}\left(\dfrac{2}{e^2}\right)^n = \dfrac{1}{1-\frac{2}{e^2}} \simeq 1.371 \quad \left(G.S., \left|\frac{2}{e^2}\right| < 1\right)$

$\sum_{n=0}^{\infty}\left(\dfrac{1}{e^2}\right)^2 = \dfrac{1}{1-\frac{1}{e^2}} = 1.1565 \quad \left(G.S., \left|\frac{1}{e^2}\right| < 1\right)$

$\therefore \sum_{n=0}^{\infty} e^{-2n}(2^n - 1) = 1.3711 - 1.1565 = \boxed{0.2146}$

b) $\sum_{n=1}^{\infty}\left(\tan n - \tan(n-1)\right)$

Sol: The series is telescoping,

$S_1 = \tan 1 - \cancel{\tan 0}^{\,0}$, $S_2 = (\cancel{\tan 1} - \tan 0) + (\tan 2 - \cancel{\tan 1}) = \tan 2 - \cancel{\tan 0}^{\,0}$

$S_3 = (\cancel{\tan 1} - \tan 0) + (\cancel{\tan 2} - \cancel{\tan 1}) + (\tan 3 - \cancel{\tan 2}) = \tan 3 - \cancel{\tan 0}^{\,0}$

$\vdots$

$S_n = (\cancel{\tan 1} - \tan 0) + (\cancel{\tan 2} - \cancel{\tan 1}) + \cdots + (\tan n - \cancel{\tan(n-1)}) = \tan n - \cancel{\tan 0}^{\,0}$

$\therefore \quad S_n = \tan n$

لاحظ أن الدالة tan n تأخذ كل القيم على R على أي فترة طولها π / لذلك لا يمكن أن يكون
هناك رقم L في R يكون نهاية ل tan n عندما n → ∞ لذا

$\lim S_n$ d.n.e. and hence the series diverges.

3) For what values of a, the G.S. converges and what is the sum

$$\sum_{n=0}^{\infty} (\sin a)^n = 1 + \sin a + \sin^2 a + \cdots$$

<u>Sol:</u> Clearly, the series is geometric series with $r = \sin a$, so it converges when $|\sin a| < 1$
or $a = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$ (كل المضاعفات الفردية لـ $\frac{\pi}{2}$)
since $\sin\left((2n+1)\frac{\pi}{2}\right) = \mp 1$

and the series converges to $\dfrac{1}{1-\sin a}$.

Determine whether the following series converge, converge abs., converge conditionally or div.?

4) $$\sum_{n=1}^{\infty} \frac{n + \sqrt{n} - 1}{n + n^3}$$

Sol: Take $a_n = \dfrac{n+\sqrt{n}-1}{n+n^3}$, $b_n = \dfrac{n}{n^3} = \dfrac{1}{n^2}$,

$\sum b_n = \sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$)

Consider $\lim \dfrac{a_n}{b_n} = \lim \left(\dfrac{n+\sqrt{n}-1}{n+n^3}\right) \cdot n^2 = 1$

so by LCT, $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sqrt{n}-1}{n+n^3}$ converges.

5) $$\sum_{n=1}^{\infty} \frac{n+1}{(5n+2)\sqrt{n}}$$

<u>Sol:</u> Take $a_n = \dfrac{n+1}{(5n+2)\sqrt{n}}$, $b_n = \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$

$\sum b_n$ diverges (p-series, $p = \frac{1}{2} < 1$)

Consider $\lim \frac{a_n}{b_n} = \lim \frac{n^{3/2} + \sqrt{n}}{5n^{3/2} + 2\sqrt{n}} = \frac{1}{5}$

so by LCT, $\sum \frac{n+1}{(5n+2)\sqrt{n}}$ diverges.

6) $\sum_{n=0}^{\infty} \frac{2^n + n}{(n+1)!}$

sol: Using ratio test:

$$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{2^{n+1} + (n+1)}{(n+2)!} \cdot \frac{(n+1)!}{2^n + n}$$

$$= \lim_{n\to\infty} \left(\frac{2\cdot 2^n + n + 1}{2^n + n}\right) \cdot \left(\frac{1}{n+2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2 + (n+1)/2^n}{1 + \frac{n}{2^n}}\right) \cdot \frac{1}{n+2} = \left(\frac{2+0}{1+0}\right) \cdot 0 = 0$$

since $\rho < 1 \Rightarrow$ The series converges.

7) $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{\ln n \sqrt{\ln^2 n - 1}}$

sol: Let $f(x) = \frac{1/x}{\ln x \sqrt{\ln^2 x - 1}}$,

1) Clearly f is cont. $\forall x \geq 3$.

2) $f(x) > 0 \quad \forall x \geq 3 \qquad (\ln^2 x > 1 \ \forall x \geq 3)$

3) يمكن اثبات تناقص الدالة بإثبات أن $f'<0$ ، ولكن نظراً لصعوبة وطول حساب المشتقة يمكن اثبات التناقص بالحل باستخدام التعريف :

$x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2 \quad ---- \quad (1)$

Moreover, for $x > 3$, $\ln x > 1 \Rightarrow \ln^2 x_1 - 1 < \ln^2 x_2 - 1$

and hence $\sqrt{\ln^2 x_1 - 1} < \sqrt{\ln^2 x_2 - 1}$ --- ②

$$\therefore \left(x_1 \ln x_1 \sqrt{\ln^2 x_1 - 1}\right) < \left(x_2 \ln x_2 \sqrt{\ln^2 x_2 - 1}\right)$$

$$\frac{1}{\left(x_1 \ln x_1 \sqrt{\ln^2 x_1 - 1}\right)} > \frac{1}{\left(x_2 \ln x_2 \sqrt{\ln^2 x_2 - 1}\right)}$$

or $f(x_1) > f(x_2) \Longrightarrow f$ is $\searrow$.

Consider $\displaystyle\int_3^\infty \frac{1/x \, dx}{\ln x \sqrt{\ln^2 x - 1}} = \lim_{b\to\infty} \int_3^b \frac{1/x \, dx}{\ln x \sqrt{\ln^2 x - 1}}$

$$= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} \frac{du}{u\sqrt{u^2-1}}$$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$x = 3 \longrightarrow u = \ln 3$

$x = b \longrightarrow u = \ln b$

$$= \lim_{b\to\infty} \sec^{-1}|u| \Big]_{\ln 3}^{\ln b}$$

$$= \lim_{b\to\infty} \left[\sec^{-1}|\ln b| - \sec^{-1}|\ln 3|\right] = \frac{\pi}{2} - \sec^{-1}|\ln 3|$$

$\Longrightarrow$ by integral test $\displaystyle\sum_{n=3}^\infty \frac{1/n}{\ln n \sqrt{\ln^2 n - 1}}$ converges.

8) $\displaystyle\sum_{n=1}^\infty \frac{\ln(10 + e^{2n})}{2n}$

sol: Consider $\lim a_n = \lim \dfrac{\ln(10+e^{2n})}{2n}$ $\left(\frac{\infty}{\infty}\right)$

$$\overset{L.R.}{=} \lim \left(\frac{\cancel{2} e^{2n}}{10 + e^{2n}}\right) \Big/ \cancel{2} \quad \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim \frac{2e^{2n}}{2e^{2n}} = 1 \neq 0$$

So the series diverges by nth-term test.

9) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln \sqrt{n}}$

sol: The series of abs. values $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln \sqrt{n}} = \sum \frac{2}{n \ln n}$

which is divergent by integral test (حلها سابقاً)

By AST, $\quad u_n = \dfrac{2}{n \ln n}$

1) $u_n > 0 \quad \forall\, n \geq 2$

2) $u_n' < 0$ (اخر) $\Longrightarrow u_n \searrow$.

3) $u_n \longrightarrow 0$

so it is convergent conditionally.

10) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n+1)}{n} \left(\tfrac{1}{2}\right)^n$

sol: The series of abs values is $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{n}\right) \left(\tfrac{1}{2}\right)^n$

$\rho = \lim \dfrac{a_{n+1}}{a_n} = \lim \left(\dfrac{2n+3}{n+1}\right)\left(\dfrac{n}{2n+1}\right) \dfrac{2^n}{2^{n+1}} = 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{2}$

Since $\rho < 1 \Longrightarrow$ by ratio test, the series of abs. values converges and hence the original series

$\displaystyle\sum (-1)^{n+1} \left(\frac{2n+1}{n}\right) \left(\tfrac{1}{2}\right)^n$ converges abs.

11) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2 \sqrt{n}\,(3+\cos n)} < \,??$

sol: $\sqrt{n+1} < \sqrt{n+n} = \sqrt{2}\,\sqrt{n} \quad \forall\, n \geq 1,$

and $\cos n > -1 \Longrightarrow 3 + \cos n > 3 - 1 = 2$

$\therefore \dfrac{1}{3+\cos n} < \dfrac{1}{2}$

$$\Rightarrow \quad \frac{\sqrt{n+1}}{n^2\sqrt{n}\,(3+\cos n)} < \frac{\sqrt{2}\,\cancel{\sqrt{n}}}{n^2\,\cancel{\sqrt{n}}\cdot 2} = \frac{1}{\sqrt{2}\,n^2}$$

since $\sum \frac{1}{\sqrt{2}\,n^2}$ converges (p-series $p=2>1$)

$\Rightarrow$ by DCT, $\sum \frac{\sqrt{n+1}}{n^2\sqrt{n}\,(3+\cos n)}$ converges.

12) Find the radius and the interval of conv. of the power series

$$\sum_{n=1}^{\infty} \frac{n!}{3\cdot 6\cdot 9\cdots 3n}(2x-1)^n$$

Sol: The series of abs. values $\sum_{n=1}^{\infty} \frac{n!}{3\cdot 6\cdot 9\cdots 3n}|2x-1|^n$

Using ratio test: $\rho = \lim \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim \frac{\cancel{(n+1)!}^{\,n+1}\;\cancel{|2x-1|^{n+1}}}{\cancel{3\cdot 6\cdot 9\cdots 3n}\,(3n+3)} \cdot \frac{\cancel{3\cdot 6\cdot 9\cdots 3n}}{\cancel{n!}\;\cancel{|2x-1|^n}}$$

$$= \lim\left(\frac{n+1}{3n+3}\right)|2x-1| = \frac{1}{3}|2x-1|.$$

which is convergent when $\rho<1 \Rightarrow -3<2x-1<3$

$\Rightarrow -2<2x<4 \Rightarrow -1<x<2$

so the radius of convergent

$$R = \frac{2-(-1)}{2} = \boxed{\frac{3}{2}}$$

ملحوظة هامة : لاحظ أن

$3\cdot 6\cdot 9\cdots 3n = 3^n(1\cdot 2\cdot 3\cdots n) = 3^n\, n!$

وبالتالي يمكن إعادة كتابة المتسلسلة بالصيغة

$$\sum_{n=1}^{\infty} \frac{n!}{3^n\, n!}(2x-1)^n = \sum \left(\frac{1}{3}\right)^n (2x-1)^n$$

وهي متسلسلة هندسية بالتأكيد متباعدة عند الحدود / ويمكن حلها

باستخدام صيغة المتسلسلات الهندسية ، ولكن باستخدام الحل في الأعلى يجب فحص الحدود .

<u>At $x=-1$:</u> The series is $\sum (-1)^n$ (بعد التبسيط) which is divergent.

<u>At $x=2$:</u> The series $\sum 1^n = \sum 1$ diverges so the interval of convergent is $(-1,2)$