Volumes Using Cross-Sections

الحجوم باستخدام المقاطع العرضية

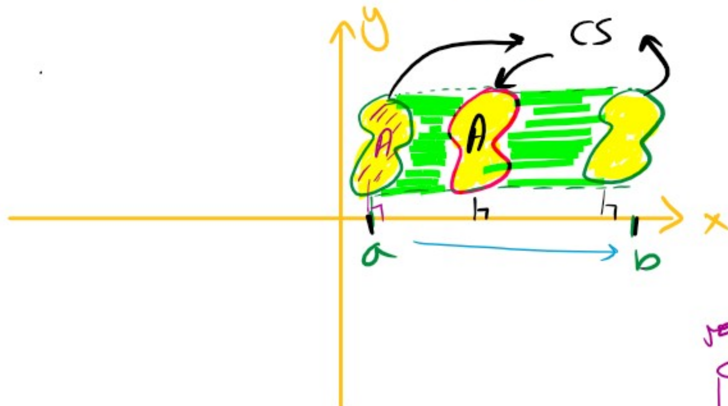
V CS

If CS $\perp$ x-axis then volume is

جمع

$$V = \int_a^b A(x)\, dx$$

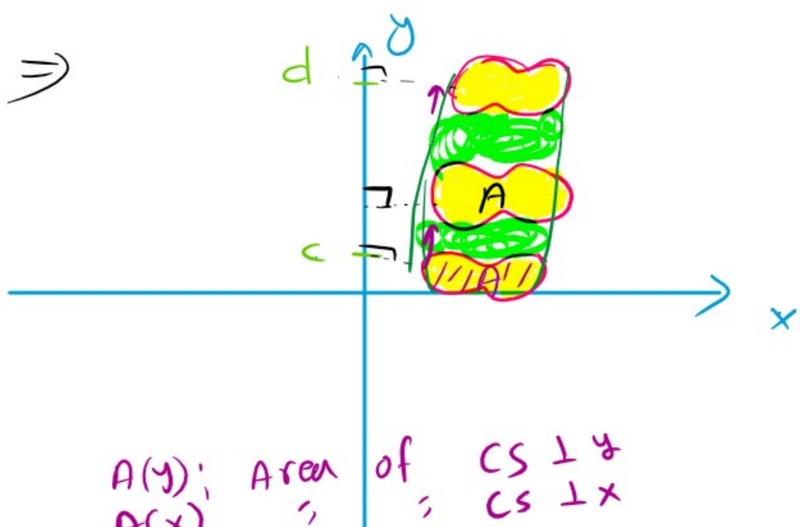
مساحة CS



If CS $\perp$ y-axis $\Rightarrow$

$$V = \int_c^d A(y)\, dy$$

مساحة CS



A(y) : Area of CS $\perp$ y
A(x) : " " CS $\perp$ x

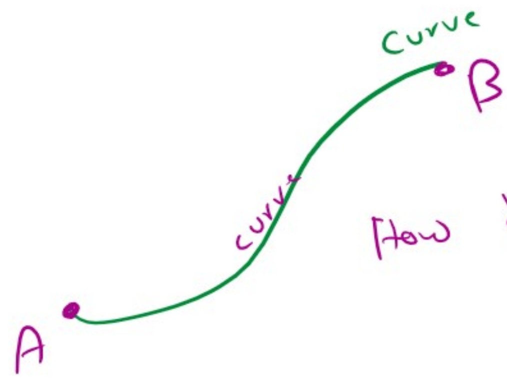
6.1 ✓ Disk Method DM
✓ Washer Method WM

→ To find V when CS $\perp$ x-axis
CS $\perp$ y-axis

6.2 Shell Method SM } → To find V

when CS ⊥ x-axis

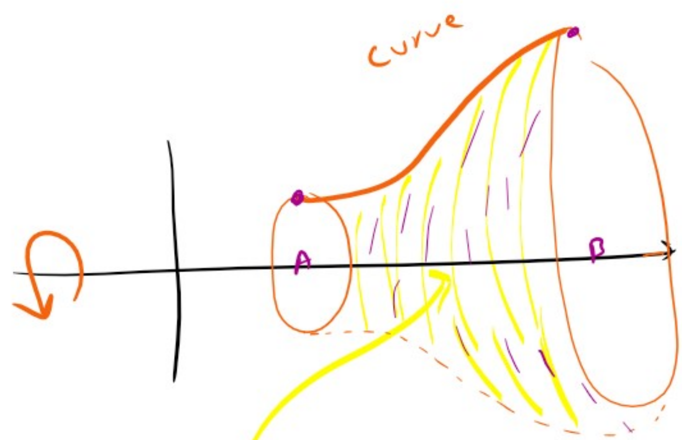
CS ⊥ y-axis



6.3

How to find length curve

6.4



✓ مساحة خارجية SA

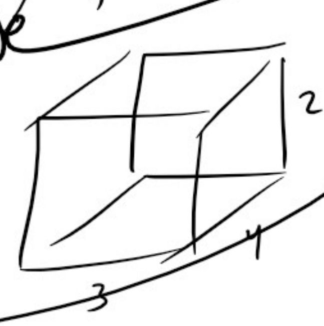
Surface Area.

أصعب إشي

شكل منتظم

V = 3x4x2

= 24

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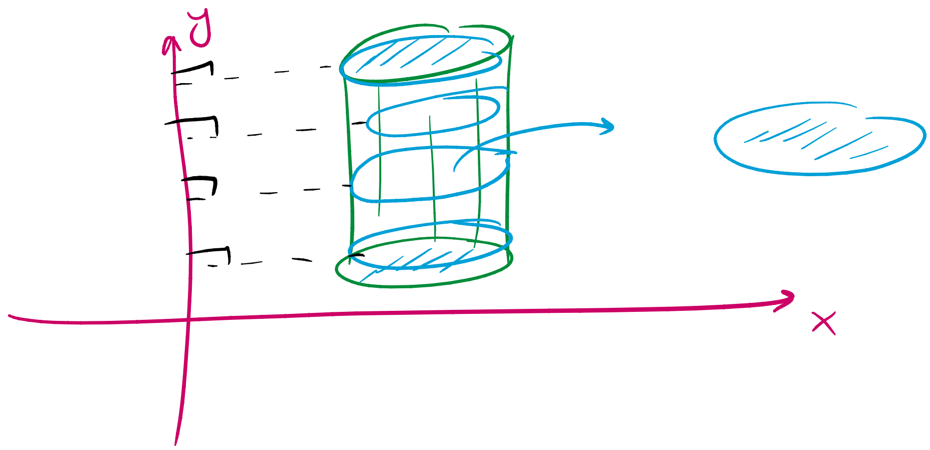
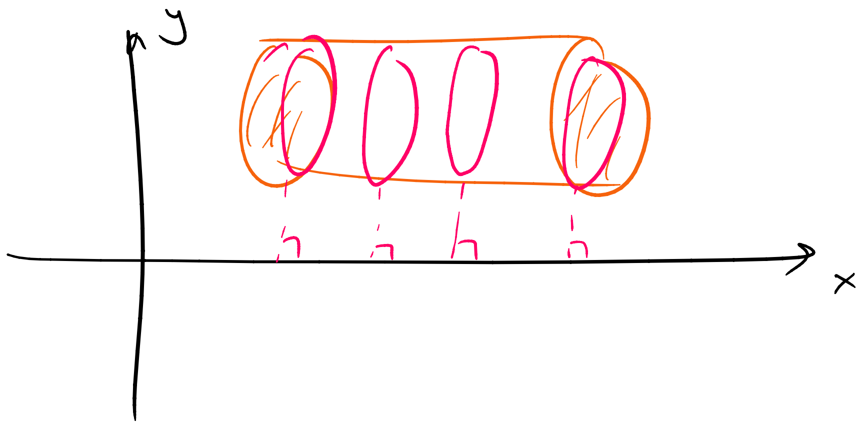

y
x


y
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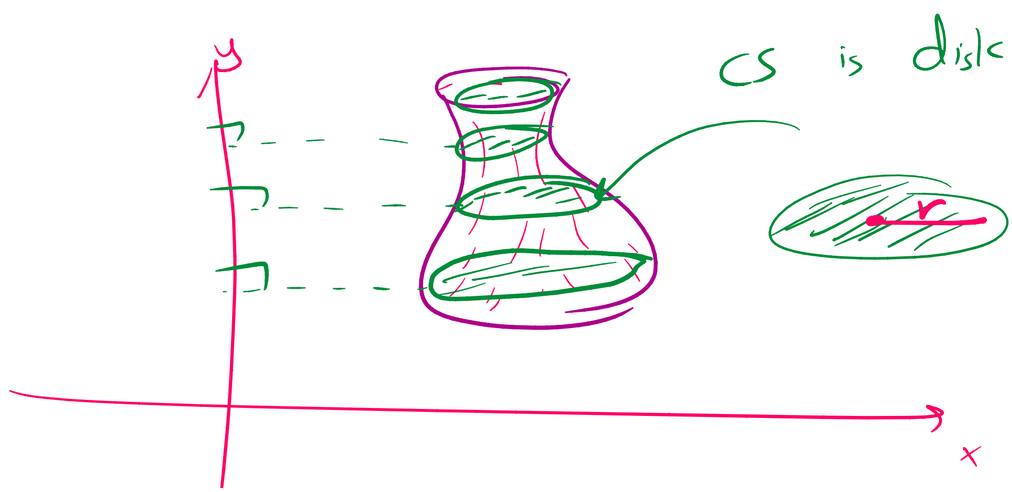
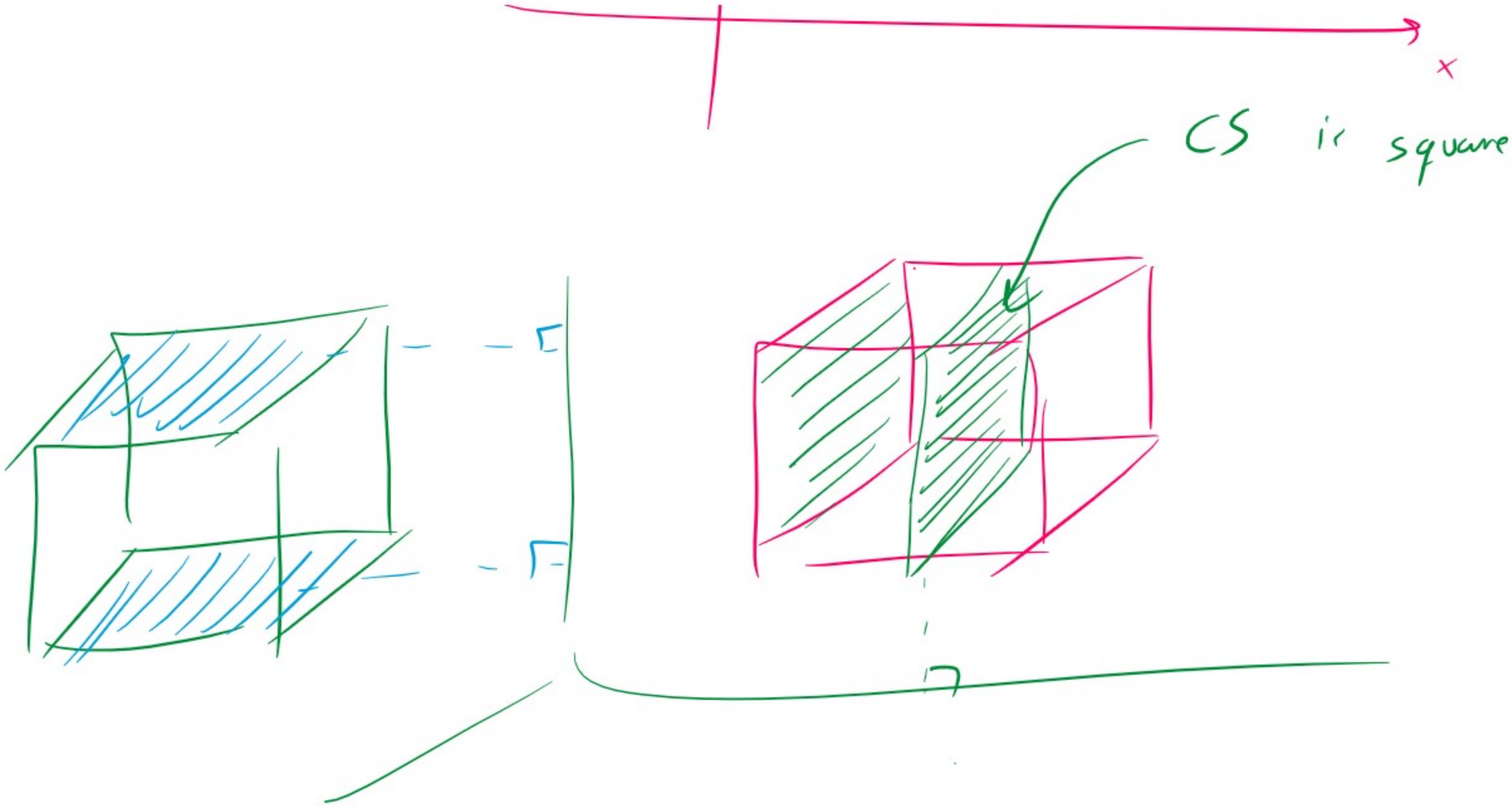
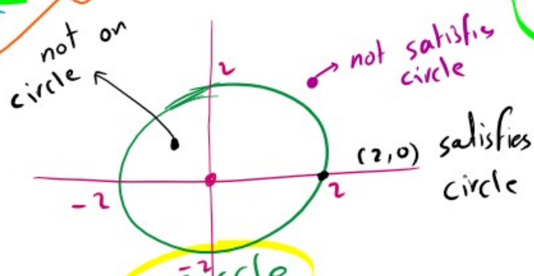

y
CS is disk
r
x



6.1 Disk Method $\Rightarrow$ CS is Disk

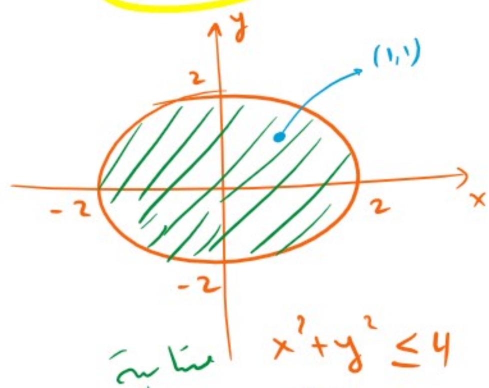


circle

$x^2+y^2=4$

$2^2+0^2=4$

$4=4$ ✓



مساحة $x^2+y^2\le 4$

Disk

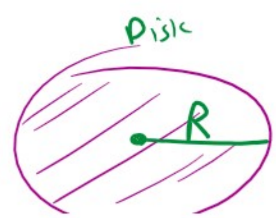
$1^2+1^2\le 4$

$2\le 4$ ✓

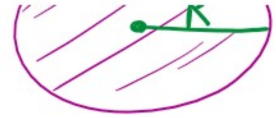
CS $\perp$ x-axis $\Rightarrow V=\int_a^b A(x)\,dx$

CS $\perp$ y-axis $\Rightarrow V=\int_c^d A(y)\,dy$

If CS is disk



cs $\perp$ y-axis $\Rightarrow V = \int_c A(y)\,dy$



$A = R^2\pi$

If CS is disk and $\perp$ x-axis $\Rightarrow V = \int_a^b \underline{A(x)}\,dx = \int_a^b \pi R^2(x)\,dx$

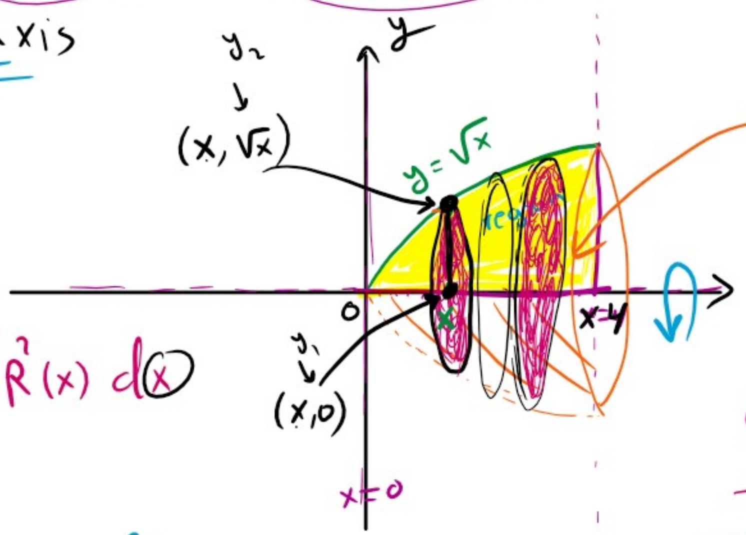
If CS is dis and $\perp$ y-axis $\Rightarrow V = \int_c^d A(y)\,dy = \int_c^d \pi R^2(y)\,dy$

Exp Find the volume of the solid generated by revolving the region bounded by

(1) the curve $y = \sqrt{x}$, $0 \le x \le 4$, x-axis

$x = 0$
$x = 4$

about x-axis



$\Downarrow$

$V = \int_a^b A(x)\,dx = \int_a^b \pi R^2(x)\,dx$

CS is disk

CS $\perp$ x-axis

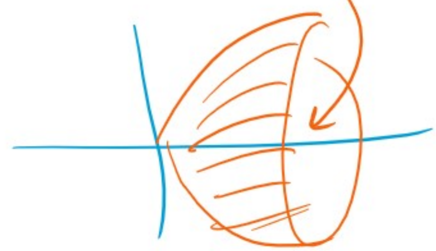
$= \int_0^4 \pi \left(\sqrt{x}\right)^2 dx$

$R = \Delta y = y_2 - y_1$
$= \sqrt{x} - 0$
$= \sqrt{x}$

$= \int_0^4 \pi x\,dx$

$$= \pi \frac{x^2}{2} \Big|_0^4 = \frac{\pi}{2}\left(4^2 - 0^2\right)$$

$$= \frac{\pi}{2}(16) = 8\pi$$



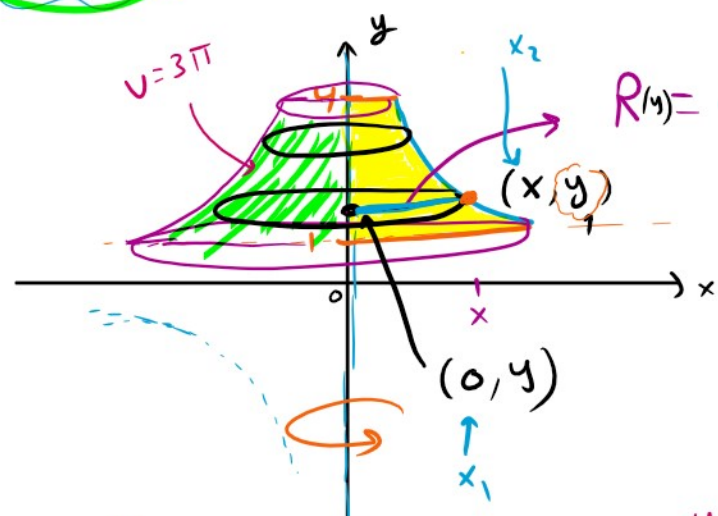
② y-axis, $x = \frac{2}{y}$, $1 \le y \le 4$ about y-axis

$y = \frac{2}{x}$

CL is disk ✓

CL ⊥ y-axis

$V = \int_c^d A(y)\,dy$



$R(y) = \Delta x = x_2 - x_1$

$= x - 0$

$= x$

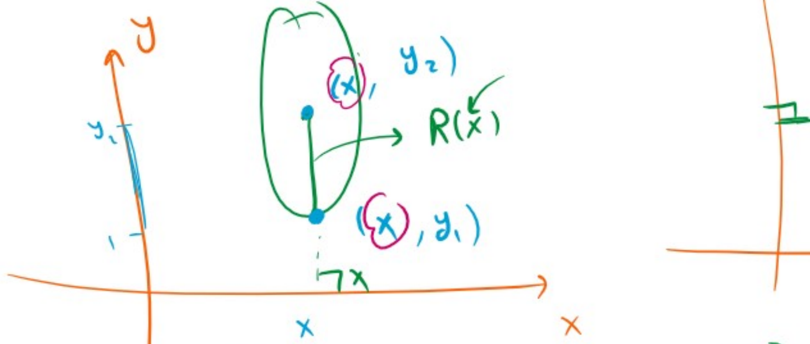
$= \frac{2}{y}$

$$= \int_c^d \pi R^2(y)\,dy = \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy = \pi \int_1^4 \frac{4}{y^2}\,dy$$

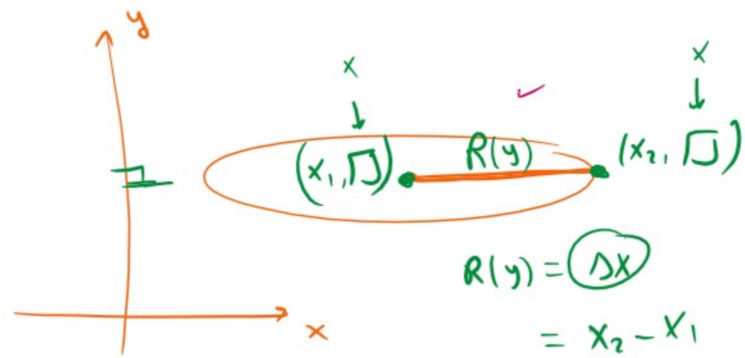
$$= 4\pi \int_1^4 y^{-2}\,dy = 4\pi \frac{y^{-1}}{-1}\Big|_1^4 = -4\pi\left[\frac{1}{y}\Big|_1^4\right]$$

$$= -4\pi\left[\frac{1}{4} - 1\right] = -\cancel{4}\pi\left(\frac{-3}{\cancel{4}}\right) = 3\pi$$

$$= -4\pi\left[\frac{1}{4} - 1\right] = -\cancel{4}\pi\left(\frac{-3}{\cancel{4}}\right) = 3\pi$$



$R(x) = \Delta y = y_2 - y_1 = \dots = \{x\}$



$R(y) = \Delta x$

$= x_2 - x_1$

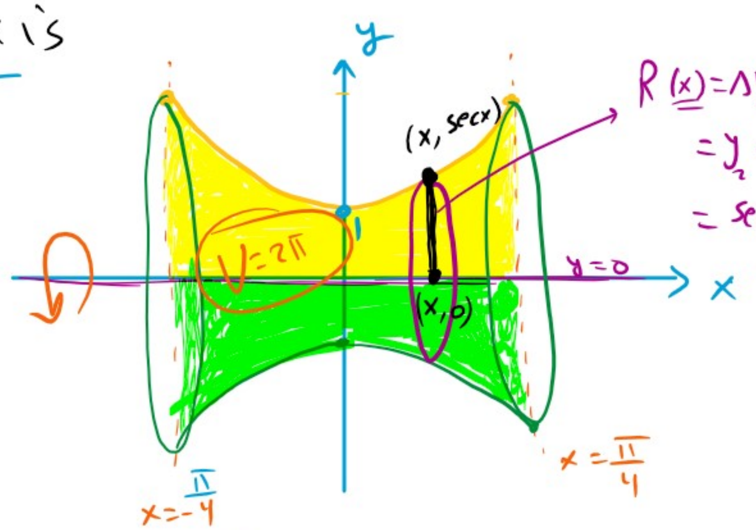
$= \dots$

$= \{y\}$

24

(3) $y = \sec x$, $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$

about x-axis



$R(x) = \Delta y$

$= y_2 - y_1$

$= \sec x - 0$

$= \sec x$

$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

CS is disk

CS $\perp$ x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi R^2(x)\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi (\sec x)^2\,dx$$

$$= \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\,dx = \pi \tan x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \pi\left[\tan\frac{\pi}{4} - \underbrace{\tan\frac{-\pi}{4}}\right]$$

tan odd

$\tan -x =$

$$\pi\left[1 \quad -1\right]$$

$= \pi \left[1 - -1 \right]$

$= 2\pi$

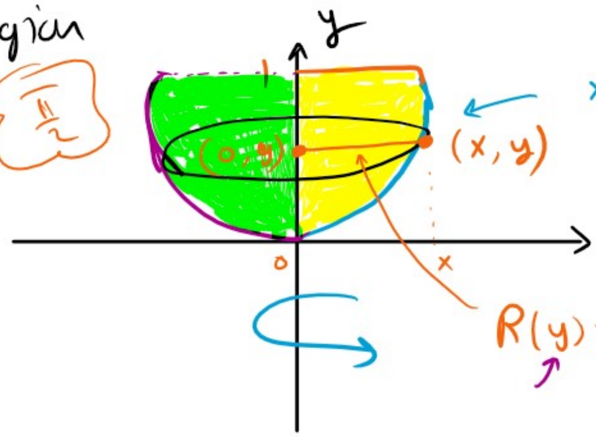
$\tan -x = -\tan x$

$\tan\frac{-\pi}{4} = -\tan\frac{\pi}{4} = -1$

Q32

4 Given the region

Find V



$\frac{\pi}{2}$

$x = \frac{\sqrt{2y}}{y^2+1}$

CS is disk

CS $\perp$ y-axis

$$V = \int_c^d \pi R^2(y)\,dy$$

$R(y) = \Delta x = x_2 - x_1 = x - 0 = x = \frac{\sqrt{2y}}{y^2+1}$

$$= \int_0^1 \pi \left(\frac{\sqrt{2y}}{y^2+1}\right)^2 dy$$

$$= \pi \int_0^1 \frac{2y}{(y^2+1)^2}\,dy$$

$u = y^2+1$

$du = 2y\,dy$

$y=0 \Rightarrow u = 0^2+1 = 1$

$y=1 \Rightarrow u = 1^2+1 = 2$

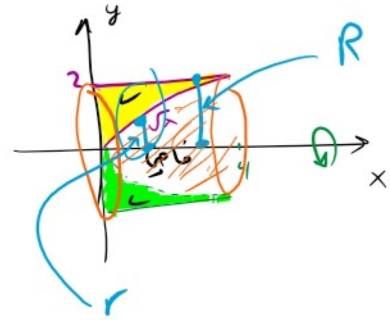
$$= \pi \int_1^2 \frac{du}{u^2}$$

$$= \pi \int_1^2 u^{-2}\,du = \pi \left.\frac{u^{-1}}{-1}\right|_1^2 = -\pi \left[\left.\frac{1}{u}\right|_1^2\right]$$

$= \left[\frac{1}{} - \frac{1}{} \right] = -\pi \left[-\frac{1}{2} \right] = \frac{\pi}{2}$

$$= -\pi\left[\frac{1}{2} - \frac{1}{1}\right] = -\pi\left[-\frac{1}{2}\right] = \frac{\pi}{2}$$

Washer Method

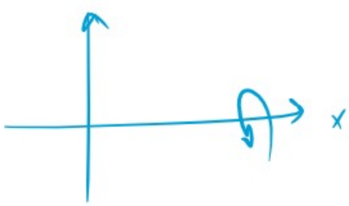


If CS ⊥ x-axis with outer radius $R(x)$ (كبير) and inner radius $r(x)$ (صغير)

Then

$$V = \int_a^b A(x)\,dx = \int_a^b \left(\pi R^2(x) - \pi r^2(x)\right) dx$$

حذف الجزء الفارغ



$$= \pi \int_a^b \left(R^2(x) - r^2(x)\right) dx$$

If CS ⊥ y-axis with $R(y)$ is outer radius and $r(y)$ is inner radius

Then

$$V = \int_c^d A(y)\,dy = \int_c^d \left(\pi R^2(y) - \pi r^2(x)\right) dy$$

$$= \pi \int_c^d \left(R^2(y) - r^2(y)\right) dy$$

c

$$\pi \int_c^d \cancel{\left(R(y) - r(y)\right)^2}$$

Exp Find the volume of the solid generated by rotating the region bounded by

① $y = 2\sqrt{x}$, $y = 2$, $x = 0$

about x-axis is

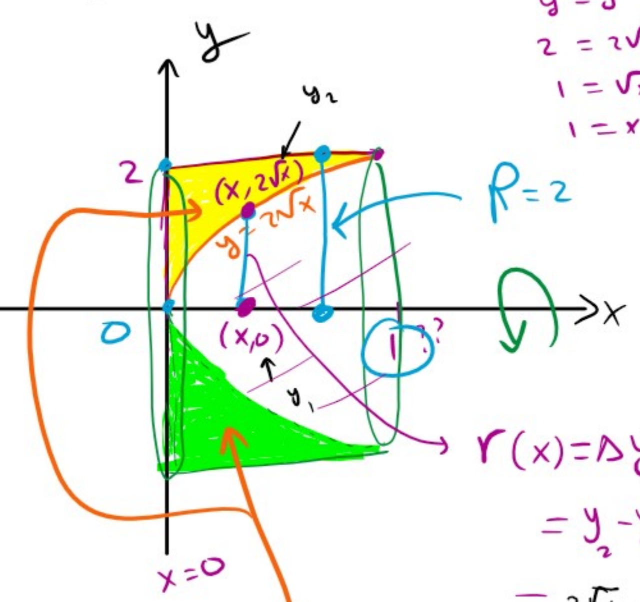
فاضي مليان → washer

$$V = \pi \int_a^b \left(R^2(x) - r^2(x)\right) dx$$

$$= \pi \int_0^1 \left[2^2 - \underbrace{(2\sqrt{x})^2} \right) dx$$

$$= \pi \int_0^1 (4 - 4x)\, dx$$

$$= \pi \left(4x - 2x^2\right)\Big|_0^1 = \pi(4-2) = 2\pi$$



$y = y$
$2 = 2\sqrt{x}$
$1 = \sqrt{x}$
$1 = x$

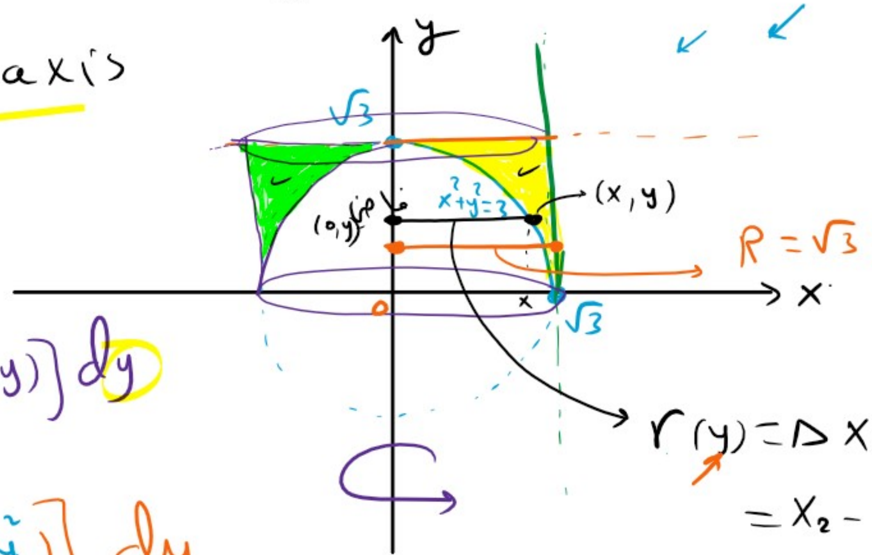
$r(x) = \Delta y$
$= y_2 - y_1$
$= 2\sqrt{x} - 0$
$= 2\sqrt{x}$

Q44 ② 2^{st} quadrant, left by the circle $x^2 + y^2 = 3$, right by line $x = \sqrt{3}$

right by line $x=\sqrt{3}$

above by line $y=\sqrt{3}$

about y-axis



washer

$$V=\pi\int_c^d\left[R^2(y)-r^2(y)\right]dy$$

$r(y)=\Delta x$

$=x_2-x_1$

$=x-0$

$=x$

$x^2+y^2=3$

$x^2=3-y^2$

$r^2(y)=x^2$

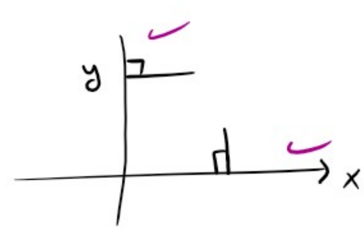
$=3-y^2$

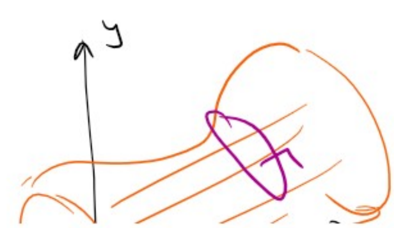
$$=\pi\int_0^{\sqrt{3}}\left[(\sqrt{3})^2-(3-y^2)\right]dy$$

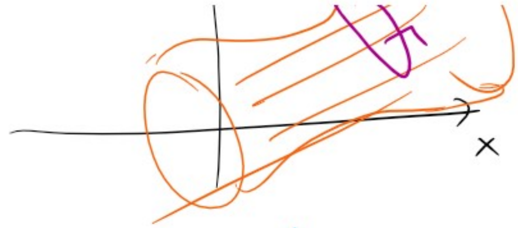
$$=\pi\int_0^{\sqrt{3}}(\cancel{3}-\cancel{3}+y^2)\,dy$$

$$=\pi\left(\frac{y^3}{3}\right)\Big|_0^{\sqrt{3}}=\frac{\pi}{3}\left(\sqrt{3}\sqrt{3}\sqrt{3}-0\right)$$

$$=\frac{\pi}{\cancel{3}}\cancel{3}\sqrt{3}=\boxed{\sqrt{3}\,\pi}$$







Remark sometimes the CS is not disk

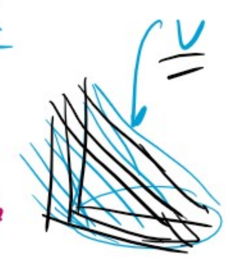
square

triangle —

$CS \perp x\text{-axis} \Rightarrow V = \int_a^b A(x)\,dx$

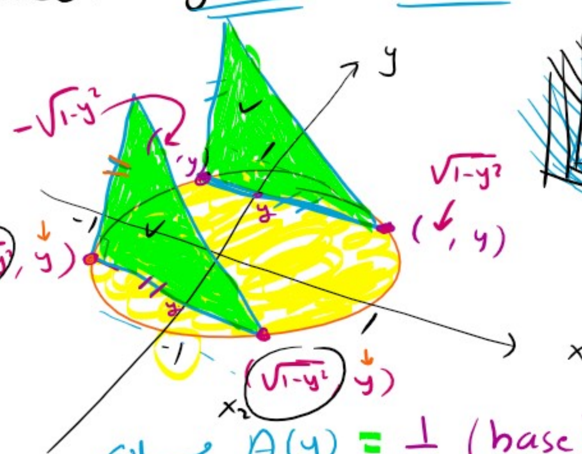
$CS \perp y\text{-axis} \Rightarrow V = \int_c^d A(y)\,dy$

Exp Find the volume of the solid whose base is the disk $x^2+y^2 \le 1$, CS's are isoscels (متساوي ساقين) right (قائمة) triangles (مثلث) with one leg on the disk, $CS \perp y\text{-axis}$ between $y=-1$, $y=1$



$CS \perp y\text{-ax} \Rightarrow V = \int_c^d A(y)\,dy$

$= \int_{-1}^{1} 2(1-y^2)\,dy$



مساحة CS $A(y) = \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}(\text{leg})(\text{leg})$

$x^2+y^2=1$

leg $= 2x$

CS [redacted]

$$x^2 + y^2 = 1$$
$$x^2 = 1 - y^2$$
$$x = \pm\sqrt{1-y^2}$$

leg $= \Delta x$
$= x_2 - x_1$
$= \sqrt{1-y^2} - (-)\sqrt{1-y^2}$
$= 2\sqrt{1-y^2}$ ✓

$= \frac{1}{2}(\text{leg})(\text{leg})$
$= \frac{1}{2}(\text{leg})^2$
$= \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2$
$= \frac{1}{\cancel{2}}\, \cancel{4}^{2}(1-y^2)$
$= \underline{2(1-y^2)}$

$$V = 2\int_{-1}^{1}(1-y^2)\,dy$$

$$= 2\left(y - \frac{y^3}{3}\right)\Big|_{-1}^{1}$$

$$= 2\left[\left(1 - \frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right)\right] = 2\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$$

$$= 2\left[2 - \frac{2}{3}\right]$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3}$$

Q2 (c)

$$y = \int_{\tan x}^{0} \frac{dt}{1+t^2} \quad . \quad \text{Find } y'$$

$u(x)$ is diff

$$y = \int^{u(x)} f(t)\,\underline{dt} \Rightarrow y' = f(u(x))\, u'(x) - \cancel{f(a)} \; 0$$

$$y = \int_{a} f(t)\,dt \implies y' = f(u(x))\,u'(x) - \cancel{f(a)}\cdot 0$$

الأول

$$y = \int_{\tan x}^{0} \frac{dt}{1+t^2} \implies y' = \frac{1}{1+0^2}(0) - \frac{1}{1+\tan^2 x}\sec^2 x$$

الثاني

$$= 0 - \frac{\sec^2 x}{1+\tan^2 x}$$

$$= -\frac{\sec^2 x}{\sec^2 x}$$

$$= -1$$

$$y = \int_{\tan x}^{0} \frac{dt}{1+t^2} = -\int_{0}^{\tan x} \frac{dt}{1+t^2}$$

$$y' = -\left[\frac{1}{1+\tan^2 x}\right]\sec^2 x = -1$$