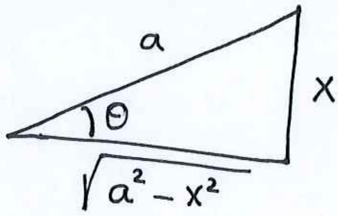
# 8.3 Trigonometric Substitution

**Case 1:** If the integrand contains $\sqrt{a^2 - x^2}$ or $(a^2 - x^2)^n$

$\longrightarrow$ Put $x = a \sin\theta$ , $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$dx = a \cos\theta \, d\theta$

* $\sin\theta = \frac{x}{a}$
* $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

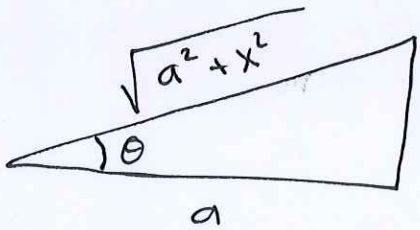



**Case 2:** If the integrand contains $\sqrt{a^2 + x^2}$ or $(a^2 + x^2)^n$

$\longrightarrow$ Put $x = a \tan\theta$ , $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = a \sec^2\theta \, d\theta$

* $\tan\theta = \frac{x}{a}$
* $\theta = \tan^{-1}\left(\frac{x}{a}\right)$





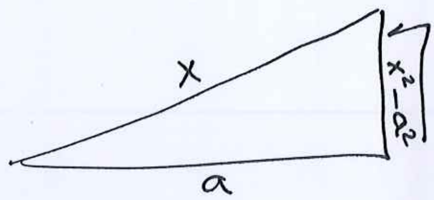



**Case 2:** If the integrand contains $\sqrt{x^2 - a^2}$ or $(x^2 - a^2)^n$

$\longrightarrow$ Put $x = a \sec\theta$ $0 \leq \theta < \frac{\pi}{2}$

$dx = a \sec\theta \tan\theta \, d\theta$

* $\sec\theta = \frac{x}{a}$
* $\theta = \sec^{-1}\left(\frac{x}{a}\right)$



Example: Evaluate the following integral

$\boxed{1}$ $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$ put $x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 3\cos\theta\, d\theta$

$$\int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\, 3\cos\theta\, d\theta$$

$$= \frac{3.(3)}{9}\int \frac{\sqrt{\cos^2\theta}\;\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta = \int \csc^2\theta - 1\, d\theta$$

$$= -\cot\theta - \theta + C$$

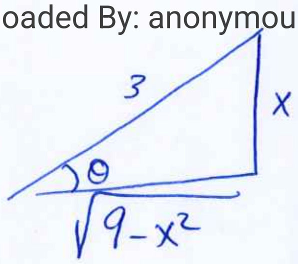
$$= \frac{-\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$





$\sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$



④. $$\int \frac{x \tan^{-1} x}{(1+x^2)^{\frac{3}{2}}}\, dx$$

put $x = \tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = \sec^2\theta \, d\theta$

$$\int \frac{\tan\theta \tan^{-1}(\tan\theta)}{(1+\tan^2\theta)^{3/2}} \sec^2\theta \, d\theta$$

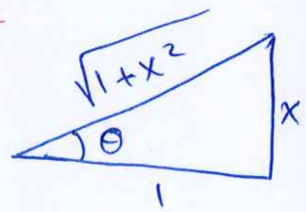
$$\int \frac{\theta \tan\theta \sec^2\theta}{\sec^3\theta}\, d\theta$$

$$\int \frac{\theta \tan\theta}{\sec\theta}\, d\theta = \int \theta \sin\theta \, d\theta$$

$$= -\theta\cos\theta + \sin\theta + c$$

| Derivative | | Integral |
|---|---|---|
| $\theta$ | $+$ | $\sin\theta$ |
| $1$ | $-$ | $-\cos\theta$ |
| $0$ | | $-\sin\theta$ |

$$= -\tan^{-1}(x) \cdot \frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + c$$




* $\tan\theta = x$

* $\theta = \tan^{-1}(x)$

(3) $\displaystyle\int \frac{\sqrt{5x^2-9}}{x}\,dx$

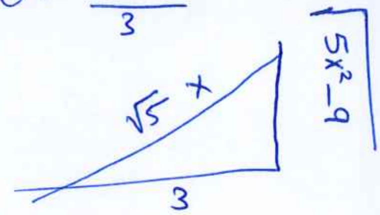
$\sqrt{5}\,x = 3\sec\theta,\quad 0 \le \theta < \frac{\pi}{2}$

$\sqrt{5}\,dx = 3\sec\theta\tan\theta\,d\theta$

$dx = \frac{3}{\sqrt{5}}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sqrt{9\sec^2\theta - 9}}{\frac{3}{\sqrt{5}}\cdot\sec\theta}\,\frac{3}{\sqrt{5}}\sec\theta\tan\theta\,d\theta$$

$\sec\theta = \frac{\sqrt{5}\,x}{3}$




$$= 3\int \tan\theta\sqrt{\sec^2\theta - 1}\,d\theta$$

$$= 3\int \tan\theta\,\tan\theta\,d\theta = 3\int \sec^2\theta - 1\,d\theta$$

$$= 3\tan\theta - 3\theta + C$$

$$= 3\,\frac{\sqrt{5x^2-9}}{3} - 3\sec^{-1}\left(\frac{\sqrt{5}\,x}{3}\right) + C$$

$$= \sqrt{5x^2-9} - 3\sec^{-1}\left(\frac{\sqrt{5}\,x}{3}\right) + C$$

$\boxed{6}$ $\displaystyle\int_1^e \frac{(\ln x)^3}{x\sqrt{1+\ln^2 x}}\,dx$

Put $\ln x = \tan\theta$, $\frac{-\pi}{2}<\theta<\frac{\pi}{2}$

$\frac{1}{x}dx = \sec^2\theta\, d\theta$

$x=1 \longrightarrow \theta=0$

$x=e \longrightarrow \theta=\frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} \frac{\tan^3\theta \sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}}$$

$$\int_0^{\frac{\pi}{4}} \frac{\tan^3\theta \sec^2\theta}{\sec\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \tan^3\theta \sec\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1) \tan\theta \sec\theta\, d\theta$$

$u = \sec\theta$

$du = \sec\theta \tan\theta\, d\theta$

$$= \int (u^2-1)\, d\theta$$

$$= \left[\frac{u^3}{3} - u\right]$$

$$= \left[\frac{\sec^3\theta}{3} - \sec\theta\right]_0^{\pi/4}$$

$$= \frac{2\sqrt{2}}{3} - \sqrt{2} - \frac{1}{3} + 1 = \frac{-1}{3}\sqrt{2} + \frac{2}{3}$$

8 $$\int \frac{x}{\sqrt{x^2-2x-3}}\,dx = \int \frac{x\,dx}{\sqrt{(x-1)^2-4}}$$

Put $x-1 = 2\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$

$x = 1+2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$= \int \frac{(1+2\sec\theta)\,2\sec\theta\tan\theta\,d\theta}{\sqrt{4\sec^2\theta-4}}$$

$$= \frac{2}{2}\int \frac{(1+2\sec\theta)\sec\theta\tan\theta}{\tan\theta}\,d\theta$$

$$= \int (1+2\sec\theta)\sec\theta\,d\theta$$

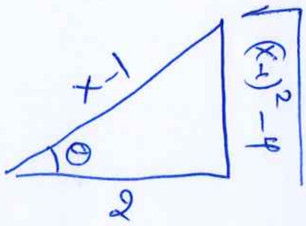
$$= \int \sec\theta + 2\sec^2\theta\,d\theta$$

$*\ \sec\theta = \frac{x-1}{2}$

$*\ \theta = \sec^{-1}\left(\frac{x-1}{2}\right)$

$$= \ln|\sec\theta+\tan\theta| + 2\tan\theta + C$$

$$= \ln\left|\frac{x-1}{2} + \frac{\sqrt{(x-1)^2-4}}{2}\right| + 2\,\frac{\sqrt{(x-1)^2-4}}{2} + C$$




$$= \ln\left|x-1+\sqrt{x^2-2x-3}\right| + \sqrt{x^2-2x-3} + C_*$$

$C_* = C - \ln 2$

(2). $$\int \frac{1}{x^3(x^2+4)}\,dx$$

$x = 2\tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{8\tan^3\theta\,(4\tan^2\theta+4)}\, 2\sec^2\theta\, d\theta$$

$$\frac{2}{8.4}\int \frac{\sec^2\theta}{\tan^3\theta\ \sec^2\theta}\, d\theta$$

$$\frac{1}{16}\int \cot^3\theta\, d\theta = \frac{1}{16}\int \cot^2\theta\,\cot\theta\, d\theta$$

$$= \frac{1}{16}\int (\csc^2\theta - 1)\cot\theta\, d\theta$$

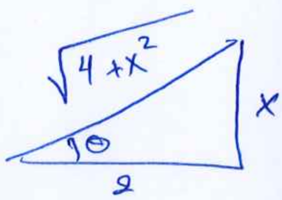
$u = \cot\theta$

$du = -\csc^2\theta\, d\theta$

$$= \frac{1}{16}\int \csc^2\theta\,\cot\theta\, d\theta - \frac{1}{16}\int \cot\theta\, d\theta$$



$$= \frac{-1}{16}\,\frac{\cot^2\theta}{2} - \frac{1}{16}\int \frac{\cos\theta}{\sin\theta}\, d\theta$$




$$= \frac{-1}{32}\cot^2\theta - \frac{1}{16}\ln|\sin\theta| + C$$

$$= \frac{-1}{32}\left(\frac{4}{x^2}\right) - \frac{1}{16}\ln\left|\frac{x}{\sqrt{4-x^2}}\right| + C$$

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7. $$\int \sqrt{\frac{4-x}{x}}\,dx = \int \frac{\sqrt{4-x}}{\sqrt{x}}\,dx$$

$y = \sqrt{x}$

$y^2 = x$

$2y\,dy = dx$

$$= \int \frac{\sqrt{4-y^2}}{y}\,2y\,dy$$

$$= 2\int \sqrt{4-y^2}\,dy$$

Put $y = 2\sin\theta,\ \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$dy = 2\cos\theta\,d\theta$

$$= 2\int \sqrt{4-4\sin^2\theta}\ 2\cos\theta\,d\theta$$

$$= 2\cdot 2\int 2\cos^2\theta\,d\theta$$

$$= 8\int \frac{1+\cos(2\theta)}{2}\,d\theta = 4\theta + \frac{4\sin(2\theta)}{2} + C$$

$$= 4\theta + 4\sin\theta\cos\theta + C$$

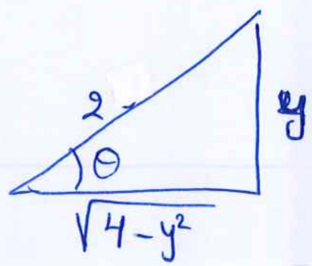




$$= 4\sin^{-1}\left(\frac{y}{2}\right) + 4\left(\frac{y}{2}\right)\left(\frac{\sqrt{4-y^2}}{2}\right) + C$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{x}\,\sqrt{4-x} + C$$

* $\sin\theta = \frac{y}{2}$
* $\theta = \sin^{-1}\left(\frac{y}{2}\right)$



5. $$\int \frac{dx}{e^x\sqrt{e^{2x}-16}}$$

Put $e^x = 4\sec\theta$, $0 \le \theta < \frac{\pi}{2}$

$$e^x\,dx = 4\sec\theta\tan\theta\,d\theta$$

$$dx = \frac{4\sec\theta\tan\theta}{4\sec\theta}\,d\theta$$

$$dx = \tan\theta\,d\theta$$

$$= \int \frac{\tan\theta\,d\theta}{4\sec\theta\sqrt{16\sec^2\theta-16}}$$

$$= \int \frac{\tan\theta\,d\theta}{16\sec\theta\tan\theta} = \frac{1}{16}\int \cos\theta\,d\theta$$

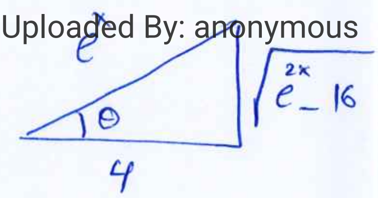
$$= \frac{1}{16}\sin\theta + C$$

$$= \frac{1}{16}\,\frac{\sqrt{e^{2x}-16}}{e^x} + C$$





* $\frac{e^x}{4} = \sec\theta$




* $\theta = \sec^{-1}\left(\frac{e^x}{4}\right)$