Chapter 3 : Some special distribution's.

3.1 : The Binomial and Relative distribution.

Def : Binomial experiment.

1. n identical trials.
2. In each trial : two Possible out comes $\left(\begin{array}{l} \text{succes } (s) \\ \text{Failure } (F) \end{array}\right)$
3. In each trial : $\left.\begin{array}{l} P(\text{success}) = p \\ P(\text{Failure}) = 1-p \end{array}\right\} \rightarrow$ remain constant.
4. All trial independent.

Define :

X : number of success in a binomial experement.

X is called binomial random variable.

RMK: X have the following p.d.f (binomial p.d.f)

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} = C_x^n \, p^x (1-p)^{n-x} & , \quad x = 0, 1, \ldots, n \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Recall : $(a+b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{m-j}$ « binomial formula »

RMK: verify that $f(x)$ is a p.d.f.

1. $f(x) \geq 0$ for all x. اعطيه

2. $\sum_x f(x) = 1$

Question : $E(x) = ?$

$$E(x) = \mu = \sum_{x} f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^{x} (1-p)^{n-x}$$

You can complete calculation OR find m.g.f if it exists.

Question: Find the m.g.f of X.

$$E(e^{tx})\, f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^{t})^{x} (1-p)^{n-x}$$

binomial formula $= (pe^{t} + 1-p)^{n} \quad , t \in \mathbb{R}$.

$$\Rightarrow \boxed{M(t) = E(e^{tx}) = (pe^{t} + 1-p)^{n} \quad , \quad t \in \mathbb{R}}.$$

Question: Find $E(x)$.

$$\dot{M}(t) = n\,(pe^{t} + 1-p)^{n-1}\,(pe^{t})$$

$$\dot{M}(0) = n\,(p.1 + 1-p)^{n-1}\,(p.1)$$

$$= np.$$

$$\Rightarrow \boxed{E(x) = \mu = \dot{M}(0) = np}.$$

Question : Find $var(x)$.

$$\bar{\bar{M}}(t) = n(pe^t + 1-p)^{n-1}(pe^t) + (pe^t)\left(n(n-1)(pe^t+1-p)^{n-2}(pe^t)\right)$$

$$\bar{\bar{M}}(0) = n(p.1 + 1-p)^{n-1}(p.1) + (p.1)\left(n(n-1)(p.1+1-p)^{n-2}(p.1)\right)$$

$$= np + n(n-1)p^2$$

$$\Rightarrow var(x) = \delta^2 = \bar{\bar{M}}(0) - (\bar{M}(0))^2$$

$$= np + n(n-1)p^2 - (np)^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np(1-p).$$

$$\Rightarrow \boxed{var(x) = \delta^2 = np(1-p)}.$$

Question : Find standard deviation.

$$\delta = \sqrt{var(x)} = \sqrt{np(1-p)}.$$

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exp 1 : let X be the number of heads (successes) in $n=7$ independent tosses of an unbiased coin. $p=\frac{1}{2}$

1, Find the p.d.f of X :

$$f(x) = \begin{cases} \binom{7}{x}\left(\frac{1}{2}\right)^x\left(1-\frac{1}{2}\right)^{7-x}, & x = 0, 1, \ldots, 7 \\ 0, & \text{elsewhere.} \end{cases}$$

2. Find m.g.f.

$M(t) = (pe^t + 1-p)^n \quad , \quad t \in \mathbb{R}.$

$M(t) = \left(\frac{1}{2}e^t + \frac{1}{2}\right)^7 \quad , \quad t \in \mathbb{R}$

3. Find $E(x)$, $var(x)$.

$E(x) = np = \frac{7}{2}$

$var(x) = np(1-p) = \frac{7}{2}\left(1-\frac{1}{2}\right) = \frac{7}{4}$

4. Find $Pr(0 \leq X \leq 1)$?

$Pr(0 \leq x \leq 1) = \sum_{x=0}^{1} f(x) = f(0) + f(1)$

$= \frac{1}{128} + \frac{7}{128} = \frac{8}{128}$

5. $Pr(X=5) = f(5) = \frac{21}{128}$.

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exp 2: If the m.g.f of a Random variable X is $M(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^5$.

1. Find p.d.f ? $n=5$, $p=\frac{1}{3}$.

$$f(x) = \begin{cases} \binom{5}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{5-x} & , \quad x = 0,1,2,3,4,5. \\ 0 & , \quad \text{elsewhere} \end{cases}$$

2. Find μ and σ^2.

$\mu = np = \frac{5}{3}$

$\sigma^2 = np(1-p) = \left(\frac{5}{3}\right)\left(\frac{2}{3}\right) = \frac{10}{9}$.

Remark: Negative Binomial distribution.

$$g(y) = \begin{cases} \binom{y+r-1}{r-1} p^r (1-p)^y & , \quad y = 0,1,2,\dots \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Remark: Geometric distribution.

$$g(y) = \begin{cases} p(1-p)^y & , \quad y = 0,1,2,\dots \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Questions:

1. verify that $g(y)$ is a p.d.f.
2. Find m.g.f if it exists. check it
3. Find $\mu = E(y)$.
4. Find $\sigma^2 = var(y)$
5. Find $\sigma = \sqrt{var(y)}$.

Recall:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\Rightarrow \quad f(x) = \frac{n!}{x!\,y!} p_1^x p_2^y.$$

$p = p_1 \quad , \quad 1-p = p_2$

$x \quad , \quad n-x = y$

Def: Trinomial distribution.

$$f(x,y)=\begin{cases} \dfrac{n!}{x!\,y!\,(n-x-y)!}\, p_1^{x}\, p_2^{y}\, (1-p_1-p_2)^{n-x-y} & , \begin{array}{l} x=0,1,\dots,n \\ y=0,1,\dots,n \\ x+y\le n \end{array} \\ 0 & , \text{elsewhere} \end{cases}$$

Remark: you can verify that $f(x,y)$ is a p.d.f. احلبو

Remark: The m.g.f of trinomial distribution:

$$M(t_1,t_2)=\left(1-p_1-p_2+p_1e^{t_1}+p_2e^{t_2}\right)^n \quad , \quad (t_1,t_2)\in\mathbb{R}^2.$$

Remark:

- The p.d.f of trinomial dist. could be written as

$$f(x,y)=\begin{cases} \dfrac{n!}{x!\,y!\,z!}\, p_1^{x}\, p_2^{y}\, p_3^{z} & , \begin{array}{l} x=0,1,2,\dots,n \\ y=0,1,2,\dots,n \\ x+y\le n \end{array} \\ 0 & , \text{elsewhere} \end{cases}$$

Where $z=n-x-y$ and $p_3=1-p_1-p_2$.

- The m.g.f of the trinomial dist. could be written as:

$$M(t_1,t_2)=\left(p_3+p_1e^{t_1}+p_2e^{t_2}\right)^n \quad , \quad (t_1,t_2)\in\mathbb{R}^2.$$

Remark:

Binomial : X x_1

Trinomial : X, Y x_1, x_2

Multinomial : – $x_1, x_2, \dots, x_{k-1}$.

Def: K-Nomial dist. (Multinomial dist.)

$$f(x_1, x_2, \dots, x_{k-1}) = \begin{cases} \dfrac{n!}{x_1!\, x_2! \cdots (n-x_1-x_2-\cdots-x_{k-1})!} p_1^{x_1} p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} (1-p_1-p_2-\cdots-p_{k-1})^{n-x_1-x_2-\cdots-x_{k-1}} & , x_1 = 0,1,\dots,n \\ & x_2 = 0,1,\dots,n \\ & x_{k-1} = 0,1,\dots,n. \\ & x_1 + \cdots + x_{k-1} \le n. \\ 0 & , \text{elsewhere}. \end{cases}$$

Remark: you can verify that $f(x_1, \dots, x_{k-1})$ is a p.d.f.

Remark: The m.g.f is given by:

$$M(t_1, \dots, t_{k-1}) = \left(1 - p_1 - p_2 - \cdots - p_{k-1} + p_1 e^{t_1} + \cdots + p_{k-1} e^{t_{k-1}}\right)^n \quad , \quad (t_1, \dots, t_{k-1}) \in \mathbb{R}^{k-1}.$$

Remark: we have trinomial dist. X, Y have the following m.g.f:

$$M(t_1, t_2) = \left(1 - p_1 - p_2 + p_1 e^{t_1} + p_2 e^{t_2}\right)^n \quad , \quad (t_1, t_2) \in \mathbb{R}.$$

$$M_1(t_1) = \left(1 - p_1 - p_2 + p_1 e^{t_1} + p_2 e^{0}\right)^n \quad , \quad t_1 \in \mathbb{R}$$

$$M_1(t_1) = \left(1 - p_1 + p_1 e^{t_1}\right)^n \quad , \quad t_1 \in \mathbb{R}.$$

$X \sim b(n, p_1)$ $\quad E(X) = n p_1 = \mu_1$

$var(X) = n p_1 (1 - p_1) = \sigma_1^2$

$M_2(t_2) = (1-P_2 + P_2 e^{t_2})^n$, $t_2 \in \mathbb{R}$

$y \sim b(n, P_2)$. $E(y) = nP_2 = \mu_2$

$Var(y) = nP_2(1-P_2) = \sigma_2^2$.

Remark: we have trinomial distribution.

$$f_{2|1}(y, x) = \begin{cases} \frac{(n-x)!}{y!(n-x-y)!} \left(\frac{P_2}{1-P_1}\right)^y \left(\frac{P_3}{1-P_1}\right)^{n-x-y} & , y = 0, 1, \dots, n-x \\ 0 & , \text{elsewhere} \end{cases}$$

$y|x = x \sim b\left(n-x, \frac{P_2}{1-P_1}\right)$

$E(y|X=x) = (n-x)\left(\frac{P_2}{1-P_1}\right)$

similarly we can find $f_{1|2}(x|y)$ and $E(x|y=y)$.

$f_{1|2}(x|y) = \begin{cases} \\ \end{cases}$

$x|y \sim b\left(n-y, \frac{P_1}{1-P_2}\right)$

$E(X|Y=y) = (n-y)\left(\frac{P_1}{1-P_2}\right)$

We can also find : $\rho = -\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}$.

Summary :

$X \sim b(n, p)$

→ $f(x) = \begin{cases} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} & , x = 0, 1, \dots, n \\ 0 & , \text{elsewhere} \end{cases}$

→ $M(t) = (1 - p + pe^t)^n \quad , t \in \mathbb{R}$.

→ $E(x) = \mu = np$

→ $var(x) = \sigma^2 = np(1-p)$.

→ $\sigma = \sqrt{var(x)} = \sqrt{np(1-p)}$.