Linear Transformation 4.1

Def A mapping from a vector space V into a vector space W (denoted by $L: V \to W$) is a linear transformation if

$$L(\alpha \vec{v}_1 + \beta \vec{v}_2) = \alpha L(\vec{v}_1) + \beta L(\vec{v}_2)$$

for all $\vec{v}_1, \vec{v}_2 \in V$ and all scalars α and β

Note: In case where the vector spaces V and W are the same, the linear transformation $L: V \to V$ is called linear operator.

Exp Let $L(x) = 3x \quad \forall x \in \mathbb{R}^2$ be an operator.

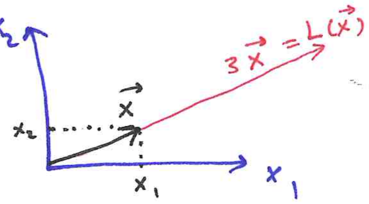
① Is $L(x)$ linear?

Yes $L(x)$ is linear operator since

$$L(\alpha x + \beta y) = 3(\alpha x + \beta y) = \alpha(3x) + \beta(3y)$$
$$= \alpha L(x) + \beta L(y)$$

② Describe geometrically the effect of the linear transformation

L is stretching by a factor of 3



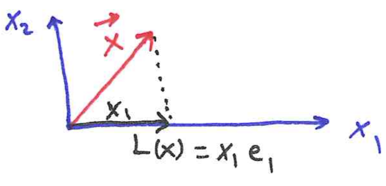
Exp ① Is the mapping $L(x) = x_1 e_1 \quad \forall x \in \mathbb{R}^2$ is a linear operator?

$$L(\alpha x + \beta y) = (\alpha x_1 + \beta y_1)\begin{pmatrix}1\\0\end{pmatrix} = \alpha\begin{pmatrix}x_1\\0\end{pmatrix} + \beta\begin{pmatrix}y_1\\0\end{pmatrix}$$
$$= \alpha x_1 e_1 + \beta y_1 e_1$$
$$= \alpha L(x) + \beta L(y)$$

Yes it is a linear operator

② Describe geometrically the effect of the linear transformation.

L is a projection onto the x_1- axis



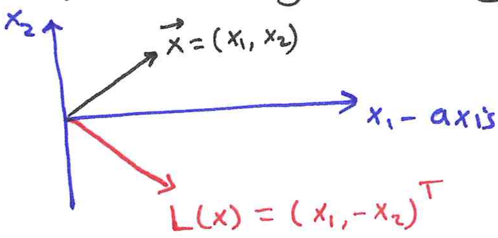
Exp ① Is the operator $L(x) = (x_1, -x_2)^T \quad \forall\, x \in \mathbb{R}^2$ linear ?

Yes L is linear operator since

$$L(\alpha x + \beta y) = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ -(\alpha x_2 + \beta y_2) \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix}$$
$$= \alpha L(x) + \beta L(y)$$

② Describe the effect of the linear transformation geometrically.

L reflects vectors about the x_1- axis

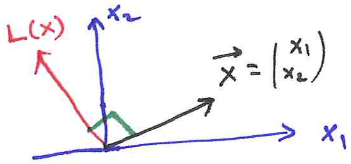


Exp $L(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \forall\, x \in \mathbb{R}^2$

① is a linear operator since

$$L(\alpha x + \beta y) = \begin{pmatrix} -(\alpha x_2 + \beta y_2) \\ \alpha x_1 + \beta y_1 \end{pmatrix} = \alpha \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + \beta \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$
$$= \alpha L(x) + \beta L(y)$$

② The operator L rotates each vector in $\mathbb{R}^2$ by $90°$ in the counterclockwise direction



Exp The mapping $L : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $L(x) = x_1 + x_2$ is a linear transformation since

$$L(\alpha x + \beta y) = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2)$$

$$= \alpha(x_1 + x_2) + \beta(y_1 + y_2)$$
$$= \alpha L(x) + \beta L(y)$$

Remark $L : V \to W$ is a linear transformation iff

$$L(v_1 + v_2) = L(v_1) + L(v_2) \qquad \boxed{\alpha = \beta = 1}$$
$$L(\alpha v) = \alpha L(v) \qquad \boxed{v = v_1,\ \beta = 0}$$

V, W are vector spaces

<u>Exp</u> Is the mapping $M: \mathbb{R}^2 \longrightarrow \mathbb{R}^1$ defined by

$$M(x) = \sqrt{x_1^2 + x_2^2} \quad \text{linear ?}$$

<u>No</u> since if we take $\alpha = -1$ and $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, then

$$L(\alpha x) = L\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \sqrt{4+0} = 2 \quad \text{but}$$

$$\alpha L(x) = -L\begin{pmatrix} 2 \\ 0 \end{pmatrix} = -\sqrt{4+0} = -2$$

<u>or</u> in general:

$$L(\alpha x + \beta y) = \sqrt{(\alpha x_1 + \beta y_1)^2 + (\alpha x_2 + \beta y_2)^2}$$

$$= \sqrt{\alpha^2(x_1^2 + x_2^2) + \beta^2(y_1^2 + y_2^2) + 2\alpha\beta(x_1 y_1 + x_2 y_2)}$$

$$\neq \alpha \sqrt{x_1^2 + x_2^2} + \beta \sqrt{y_1^2 + y_2^2}$$

<u>Exp</u> The mapping $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$L(x) = \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix} \quad \text{is linear since}$$

$$L(\alpha x + \beta y) = \begin{pmatrix} \alpha x_2 + \beta y_2 \\ \alpha x_1 + \beta y_1 \\ (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix} + \beta \begin{pmatrix} y_2 \\ y_1 \\ y_1 + y_2 \end{pmatrix} = \alpha L(x) + \beta L(y)$$

- Note that $L(x) = Ax = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1} = \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix} \quad \forall x \in \mathbb{R}^2.$

<u>Remark</u> If A is any $m \times n$ matrix, we can define a linear transformation $L_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by

$$L_A(x) = Ax \quad \forall x \in \mathbb{R}^n.$$

L_A is linear since

$$L_A(\alpha x + \beta y) = A(\alpha x + \beta y)$$
$$= \alpha Ax + \beta Ay$$
$$= \alpha L_A(x) + \beta L_A(y)$$

Remark. Let V and W be a vector spaces.
If $L: V \to W$ is a linear transformation, then

1. $L(0_v) = 0_w$ (it follows from $L(\alpha v) = \alpha L(v)$ with $\alpha = 0$)
2. if $v_1, \dots, v_n$ are elements of V and $\alpha_1, \dots, \alpha_n$ are scalars, then
$$L(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_1 L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_n L(v_n)$$
by induction
3. $L(-v) = -L(v) \quad \forall\, v \in V.$

$0_w = L(0_v) = L(v + (-v)) = L(v) + L(-v)$

Therefore, $L(-v)$ is the additive inverse of $L(v)$.

Def • If V is any vector space, then the idenity operator I is defined by $I(v) = v \quad \forall\, v \in V.$

- Clearly I is a linear transformation maps $I: V \to V$
since $I(\alpha v_1 + \beta v_2) = \alpha v_1 + \beta v_2 = \alpha I(v_1) + \beta I(v_2)$

Exp Let $L: C[a,b] \to \mathbb{R}^1$ defined by
$L(f) = \int_a^b f(x)\,dx$. Show that L is linear.

$$L(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$
$$= \alpha L(f) + \beta L(g)$$

Exp let $D: C'[a,b] \to C[a,b]$ defined by
$D(f) = f'$. Show that D is linear transformation.

$$D(\alpha f + \beta g) = (\alpha f + \beta g)' = \alpha f' + \beta g' = \alpha D(f) + \beta D(g)$$

Def [1] Let $L: V \to W$ be a linear transformation.
The Kernel of L, denoted by $\ker(L)$, is defined by

$$\ker(L) = \{ v \in V \mid L(v) = 0_w \}$$

2 • Let $L: V \to W$ be a linear transformation.
Let S be a subspace of V.
The image of S, denoted by $L(S)$, is defined by

$$L(S) = \{ w \in W \mid w = L(v) \text{ for some } v \in S \}$$

• The image of the entire vector space, $L(V)$, is called the range of L

Th 4.1 Let $L: V \to W$ be a linear transformation. Let S be a subspace of V. Then

(a) $\ker(L)$ is a subspace of V

(b) $L(S)$ is a subspace of W.

Proof: (a) • $\ker(L)$ is nonempty since $0_V \in \ker(L)$

• let $\vec{v} \in \ker(L)$ be arbitrary and α be a scalar.
Then $L(\alpha \vec{v}) = \alpha L(\vec{v}) = \alpha 0_W = 0_W \Rightarrow \alpha \vec{v} \in \ker(L)$

• Let $v_1, v_2 \in \ker(L)$ be arbitrary.
$L(v_1 + v_2) = L(v_1) + L(v_2) = 0_W + 0_W = 0_W \Rightarrow v_1 + v_2 \in \ker(L)$

(b) • $L(S)$ is nonempty since $0_W = L(0_V) \in L(S)$.

• Let $w \in L(S)$, then $w = L(v)$ for some $v \in S$.
If α is any scalar, then
$\alpha w = \alpha L(v) = L(\alpha v)$ but $\alpha v \in S$ since S is a subspace. Hence, $\alpha w \in L(S)$

• Let $w_1, w_2 \in L(S)$ be arbitrary $\Rightarrow \exists\, v_1, v_2 \in S$ s.t $L(v_1) = w_1$ and $L(v_2) = w_2$. Thus,
$w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$ but $v_1 + v_2 \in S$ since S is a subspace, hence, $w_1 + w_2 \in L(S)$.

Exp Consider the linear operator on $\mathbb{R}^2$ defined by

$$L(x) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

① Find $Ker(L) = Span(e_2)$

Note that $Ker(L)$ is a subspace of $\mathbb{R}^2$ with dimenision 1

② Range of L : $L(\mathbb{R}^2) = span(e_1)$

Note that $L(\mathbb{R}^2)$ is a subspace of $\mathbb{R}^2$ with dimension 1

Exp Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$L(x) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$$

① Find $Ker(L)$: If $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in Ker(L)$, then

$$\begin{matrix} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right]$$

$x_3 = \alpha$ is free $\Rightarrow x_2 = -\alpha$, $x_1 = \alpha$

$$Ker(L) = \left\{ x \in \mathbb{R}^3 : x = \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$= span\left((1, -1, 1)^T \right)$$

Note that $Ker(L)$ is subspace of $\mathbb{R}^3$ with dimension 1

② Let S be the subspace of $\mathbb{R}^3$ spanned by e_1 and e_3

Find the range of S.

- If $x \in S$, then $x = \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix} \Rightarrow L(x) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

  $L(S) = \mathbb{R}^2$

- This implies that $L(\mathbb{R}^3) = \mathbb{R}^2$

Exp let $D : P_3 \to P_2$ be the differentiation operator defined by $D(p(x)) = p'(x)$. Find $Ker(D) = P_1$ = all polynomials of degree 0

- Find the range of D : $D(P_3) = P_2$
