Exercises of chapter 15 :

Q1: prove Theorem 15.1 .

let ϕ be a Ring Homomorphism from a Ring R to a Ring S. let A be a subring of R and let B an ideal of S.

1. For any $r \in R$ and any positive integer n, $\phi(nr) = n\phi(r)$ and $\phi(r^n) = (\phi(r))^n$.

proof :

$\rightarrow \phi(r^n) = (\phi(r))^n \quad , n \in \mathbb{Z}$

① By induction : let $n \in \mathbb{Z}^+$

① $n=1 \rightarrow \phi(r^1) = \phi(r)$

② $n=k \rightarrow$ suppose $\phi(r^k) = (\phi(r))^k$

③ $n=k+1 \rightarrow \phi(r^{k+1}) = \phi(r^k \cdot r)$

$= \phi(r^k)\ \phi(r)$

$= (\phi(r))^k\ \phi(r)$

$= (\phi(r))^{k+1}$

② $n=0 \rightarrow r^0 = e$

$\phi(r^0) = \phi(e) = e = (\phi(r))^0$.

③ $n \in \mathbb{Z}^-$

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Q2: Prove Theorem 15.2. "on Note"

" let ϕ be a Ring homomorphism from a Ring R to a Ring S. then $Ker\phi \trianglelefteq R$.

Proof:

→ $0 \in Ker\phi$ since $\phi(0) = 0$, $Ker\phi$ nonempty.

→ let $x, y \in Ker\phi \Rightarrow \phi(x) = \phi(y) = 0$

Now $\phi(x-y) = \phi(x) - \phi(y)$ since ϕ Ring hom.

$= 0 - 0$

$= 0$

$\Rightarrow (x-y) \in Ker\phi$.

→ let $x \in Ker\phi$ and $r \in R \Rightarrow \phi(x) = 0$

Now $\phi(rx) = \phi(r)\,\phi(x)$ since ϕ Ring homo.

$= \phi(r) \cdot 0$

$= 0$

$\Rightarrow rx \in R$.

and $\phi(xr) = \phi(x)\,\phi(r)$ since ϕ Ring homo.

$= 0 \cdot \phi(r)$

$= 0$

so $xr \in R$.

So By 1+2+3 we proved $Ker\phi \trianglelefteq R$.

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Q3: prove Theorem 15.3.

" let ϕ be a Ring homom. from R to S. then the mapping from $\underbrace{R|Ker\,\phi \text{ to } \phi(R)}_{\psi}$ given by $r + Ker\,\phi \longrightarrow \phi(r)$ is an isomorphism.

($R|Ker\,\phi \cong \phi(R)$).

proof :

① ψ is 1-1 since, suppose $\psi(r_1 + Ker\,\phi) = \psi(r_2 + Ker\,\phi)$

$\Rightarrow \phi(r_1) = \phi(r_2)$

$\Rightarrow \phi(r_1 - r_2) = 0$

$\Rightarrow r_1 - r_2 = 0$

$\Rightarrow r_1 - r_2 \in Ker\,\phi$

$\Rightarrow r_1 + Ker\,\phi = r_2 + Ker\,\phi$

② ϕ is onto since, if $s \in \phi(R) \Rightarrow \exists r \in R$ s.t $\phi(r) = s$

$\Rightarrow \psi(r + Ker\,\phi) = \phi(r) = s$.

Now, $\psi(r_1 + Ker\,\phi + r_2 + Ker\,\phi) = \psi(r_1 + r_2 + Ker\,\phi)$.

$= \phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$

$= \psi(r_1 + Ker\,\phi) + \psi(r_2 + Ker\,\phi)$.

also, $\psi(r_1 + Ker\,\phi \cdot r_2 + Ker\,\phi) = \psi(r_1 r_2 + Ker\,\phi)$

$= \phi(r_1 r_2)$

$= \phi(r_1)\,\phi(r_2)$

$= \psi(r_1 + Ker\,\phi) \cdot \psi(r_2 + Ker\,\phi)$.

so ψ is isomorphisim

Q4: Prove Theorem 15.4.

" every ideal of a Ring R is the Kernal of a Ring homo. of R "

in particular, an ideal A is the Kernal of the mapping $r \to r+A$ from R to $R|A$.

Proof:

let A be an ideall of R then

$$\phi : R \to R|A$$
$$r \to r+A$$ is a Ring homo. of R.

since $\rightsquigarrow$ $\phi(r_1+r_2) = (r_1+r_2)+A$

$= r_1+A + r_2+A$

$= \phi(r_1) + \phi(r_2)$

also $\rightsquigarrow$ $\phi(r_1 r_2) = r_1 r_2 + A$

$= (r_1+A) \cdot (r_2+A)$

$= \phi(r_1) \cdot \phi(r_2)$.

and $\rightarrow$ $Ker\phi = \{ r \in R : \phi(r) = 0+A \}$

$= \{ r \in R : r \in A \}$

$= A$.

Q5: Show that the correspondence $x \to 5x$ from Z_5 to Z_{10} does Not preserve addition.

$$\phi(2+4) = \phi(\underset{\text{on } Z_5}{\underline{6}})$$
$$= \phi(1)$$
$$= 5$$

But $\phi(2) + \phi(4) = \underline{10 + 20}$ on Z_{10}
$$= 0$$

$$\phi(2+4) \neq \phi(2) + \phi(4).$$

Q6: Show that the correspondence $x \to 3x$ from $Z_4 \to Z_{12}$ does not preserve multiplication.

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$\boxed{+ Q7}$ ما شرح النظرية

Q8: Prove that every Ring homomorphism ϕ from Z_n to itself has the form $\phi(x) = ax$, where $a^2 = a$.

let $\phi : Z_n \to Z_n$ be a Ring homomorphism .

Set $a = \phi(1)$, Then

for any $m \in Z_n$ we have $\phi(m) = \phi(m \cdot 1)$

$= m\,\phi(1)$

$\phi(m) = m a$

Now $a = \phi(1) = \phi(1.1)$

$= \phi(1)\,\phi(1)$

$= a^2$

Thus, we have shown that $\phi(x) = ax \quad \forall x \in Z_n$ with $a^2 = a$.

Q9: suppose that ϕ is a Ring homomorphism from Z_m to Z_n . Prove that if $\phi(1) = a$ then $a^2 = a$. the converse is false .

let $\phi : Z_m \to Z_n$ be a Ring homo .

set $a = \phi(1)$

$= \phi(1.1)$

$= \phi(1)\,\phi(1)$

$= a \cdot a$

$= a^2$ $\Rightarrow$ so $a = a^2$

Counter exp of the converse :

$\phi : Z \to Z_6$, $\phi : X \to X$

let $t = 3 \bmod 6$ an element in ϕ

$\to t^2 = 9 \bmod 6 = 3$ $\Big\} \to t = t^2$

But $\phi(1) = 1 \bmod 6$, $\phi(1) \neq t$

