45

HW #3

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Math 342B Assignment ~~5~~

Problems, Section 12.1
Solve the following differential equations by series and also by an elementary method and verify that your solutions agree. Check your results by computer.

1.
$xy' = xy + y$.

Separation of variables:
From the given equation we get

$$xy' = xy + y \implies xy' = y(x+1) \implies \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+1}{x} \implies \int \frac{1}{y}\mathrm{d}y = \int \frac{x+1}{x}\mathrm{d}x. \tag{1}$$

So, the left hand side evaluates to

$$\ln y + C_1 \tag{2}$$

for some constant $C_1 \in \mathbb{R}$ and the right hand side is simplified as follows:

$$\int \frac{x+1}{x}\mathrm{d}x = \int (1 + x^{-1})\mathrm{d}x = x + \ln x + C_2 \tag{3}$$

for some $C_2 \in \mathbb{R}$. Let $C = C_2 - C_1$. Then we have from (1), (2), and (3) that

$$\ln y = x + \ln x + C, \tag{4}$$

so

$$\begin{aligned} y &= \exp(x + \ln x + C) \\ &= \exp(x)\exp(\ln x)\exp(C) \\ &= y_0 x \mathrm{e}^x, \end{aligned} \tag{5}$$

where $y_0 = \exp(C)$. So, $y = y_0 x \mathrm{e}^x$. Wolfram Alpha confirms this.

Series method:
Assume a solution

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{6}$$

for $a_n \in \mathbb{R}$. Then

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}. \tag{7}$$

Plugging (6) and (7) into the differential equation, we get

$$xy' = xy + y \implies x\sum_{n=0}^{\infty} a_n n x^{n-1} = (x+1)\sum_{n=0}^{\infty} a_n x^n. \tag{8}$$

So,

$$\sum_{n=0}^{\infty} a_n n x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} \tag{9}$$
$$\implies \sum_{n=0}^{\infty} a_{n+1}(n+1)x^{n+1} = a_0 + \sum_{n=0}^{\infty} (a_{n+1} + a_n)x^{n+1}.$$

So, $a_0 = 0$ and

$$a_{n+1}(n+1) = a_{n+1} + a_n \implies a_{n+1} = \frac{1}{n}a_n. \tag{10}$$

So, $a_2 = a_1$, $a_3 = \frac{1}{2}a_2 = \frac{1}{2}a_1$, $a_4 = \frac{1}{3}a_3 = \frac{1}{6}a_1$. In general, for $n \geq 1$, we have

$$a_n = \frac{1}{(n-1)!}a_1. \tag{11}$$

Therefore, our solution becomes

$$y = a_1 \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^n. \tag{12}$$

This is equivalent to $y = y_0 x \mathrm{e}^x$ if $y_0 = a_1$:

$$a_1 \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^n = a_1 x \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^{n-1} = a_1 x \sum_{n=0}^{\infty} \frac{1}{n!}x^n = a_1 x \mathrm{e}^x. \tag{13}$$

Thus, the solution to the differential equation is

$$y = a_1 \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^n = a_1 x \mathrm{e}^x \tag{14}$$

for $a_1 \in \mathbb{R}$.

2.
$y' = 3x^2 y.$

Separation of variables:
From the given equation we get

$$y' = 3x^2 y \implies \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \implies \int \frac{1}{y}\mathrm{d}y = 3\int x^2 \mathrm{d}x \implies \ln y = x^3 + C \tag{15}$$

where $C \in \mathbb{R}$ is a constant. Let $y_0 = \exp(C)$. Then we get

$$y = \exp\left(x^3 + C\right) = \exp\left(x^3\right)\exp(C) = y_o \mathrm{e}^{x^3}. \tag{16}$$

So, $y = y_0 \mathrm{e}^{x^3}$. Wolfram Alpha agrees.

Series method:
Assume a solution as in (6), so that (7) follows. Then we plug (6) and (7) into the differential equation to get

$$\sum_{n=0}^{\infty} a_n n x^{n-1} = 3x^2 \sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=0}^{\infty} a_n x^{n+2}. \tag{17}$$

So, we have

$$a_1 + 2a_2 + \sum_{n=0}^{\infty} a_{n+3}(n+3)x^{n+2} = 3\sum_{n=0}^{\infty} a_n x^{n+2}. \tag{18}$$

So, $a_1 = a_2 = 0$, a_0 is arbitrary, and

$$a_{n+3}(n+3) = 3a_n \implies a_{n+3} = \frac{3}{n+3}a_n. \tag{19}$$

So, $a_3 = \frac{3}{3}a_0 = a_0$, $a_6 = \frac{3}{6}a_3 = \frac{1}{2}a_3 = \frac{1}{2}a_0$, $a_9 = \frac{3}{9}a_6 = \frac{1}{3}a_6 = \frac{1}{6}a_0$. In general,

$$a_{3n} = \frac{1}{n!}a_0. \tag{20}$$

Thus, our solution is

$$y = a_0\left(1 + x^3 + \frac{1}{2}x^6 + \frac{1}{9}x^9 + \cdots\right) = a_0\sum_{n=0}^{\infty}\frac{1}{n!}x^{3n}. \tag{21}$$

This is exactly the Taylor expansion of $a_0e^{x^3}$. So, if $y_0 = a_0$, we have the general solution of the differential equation:

$$y = a_0\sum_{n=0}^{\infty}\frac{1}{n!}x^{3n} = a_0e^{x^3} \tag{22}$$

for $a_0 \in \mathbb{R}$.

4.
$y'' = -4y.$

Linear homogeneous second order DE:
Since the differential equation is linear, assume a solution $y = e^{nx}$. Then $y' = ne^{nx}$ and $y'' = n^2e^{nx}$. So, we get

$$y'' = -4y \implies n^2e^{nx} = -4e^{nx} \implies n^2 = -4 \implies n = \pm 2i. \tag{23}$$

Thus,

$$y = Ae^{i2x} + Be^{-i2x} \tag{24}$$

for some $A, B \in \mathbb{R}$. So, using Euler's formula, (24) becomes

$$\begin{aligned} y &= A[\cos(2x) + i\sin(2x)] + B[\cos(-2x) + i\sin(-2x)] \\ &= A[\cos(2x) + i\sin(2x)] + B[\cos(2x) - i\sin(2x)] \\ &= (A+B)\cos(2x) + i(A-B)\sin(2x). \end{aligned} \tag{25}$$

Let $C_0 = A + B$ and let $C_1 = (A-B)i$. Then $y = C_0\cos(2x) + C_1\sin(2x)$. Wolfram Alpha confirms this.

Series method:
Assume a solution as in (6), so that (7) follows. Then we plug (6) and (7) into the differential equation to get

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = -4\sum_{n=0}^{\infty} a_n x^n. \tag{26}$$

So, we have

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = -4\sum_{n=0}^{\infty} a_n x^n. \tag{27}$$

Thus, a_0 and a_1 are arbitrary and

$$a_{n+2}(n+2)(n+1) = -4a_n \implies a_{n+2} = -\frac{4}{(n+2)(n+1)}a_n. \tag{28}$$

So, $a_2 = -\frac{4}{2\cdot 1}a_0$, $a_3 = -\frac{4}{3\cdot 2}a_1$, $a_4 = -\frac{4}{4\cdot 3}a_2 = \frac{16}{4\cdot 3\cdot 2\cdot 1}a_0$, $a_5 = -\frac{4}{5\cdot 4}a_3 = \frac{16}{5\cdot 4\cdot 3\cdot 2}a_1$. Thus, in general we have

$$a_{2n} = \frac{(-1)^n 4^n}{(2n)!}a_0, \qquad n \geq 0, \tag{29}$$

and

$$a_{2n+1} = \frac{(-1)^n 4^n}{(2n+1)!}a_1, \qquad n \geq 0. \tag{30}$$

So, the solution becomes

$$\begin{aligned} y &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n+1)!}x^{2n+1} \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} + \frac{a_1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} \\ &= a_0 \cos(2x) + \frac{a_1}{2}\sin(2x). \end{aligned} \tag{31}$$

If we let $a_0 = C_0$ and $\frac{a_1}{2} = C_1$ then (31) is identical to the original answer of $y = C_0\cos(2x) + C_1\sin(2x)$. Thus, the general solution of the differential equation is

$$y = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} + C_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = C_0\cos(2x) + C_1\sin(2x). \tag{32}$$

9.
$(x^2+1)y'' - 2xy' + 2y = 0.$

Reduction of order:
Given the differential equation $(x^2+1)y'' - 2xy' + 2y = 0$, assume a solution $y_1 = x^m$ for some $m \in \mathbb{R}$. Then $y_1' = mx^{m-1}$ and $y_1'' = m(m-1)x^{m-2}$, so we get

$$\begin{aligned} &(x^2+1)y_1'' - 2xy_1' + 2y_1 = 0 \implies m(m-1)\left(x^m + x^{m-2}\right) - 2mx^m + 2x^m = 0 \\ &\implies \left(m^2 - 3m + 2\right)x^m + \left(m^2 - m\right)x^{m-2} = 0 \implies m^2 - 3m + 2 = m^2 - m = 0 \\ &\implies m = 1. \end{aligned} \tag{33}$$

So $y_1 = x$ is a solution of the differential equation. Suppose $y_2 = y_1(x)v(x) = xv(x)$ is another solution to the differential equation for some function $v : \mathbb{R} \to \mathbb{R}$. Then

$y_2' = v + xv'$ and $y_2'' = 2v' + xv''$. Plugging these into the differential equation, we get

$$\begin{aligned}
(34) \qquad & (x^2+1)y_2'' - 2xy_2' + 2y_2 = 0 \\
\Longrightarrow\ & (x^2+1)(2v' + xv'') - 2x(v + xv') + 2xv = 0 \\
\Longrightarrow\ & 2x^2v' + 2v' + x^3v'' + xv'' - 2xv - 2x^2v' + 2xv = 0 \\
\Longrightarrow\ & 2v' + x^3v'' + xv'' = 0 \\
\Longrightarrow\ & (x^3 + x)v'' + 2v' = 0 \\
\Longrightarrow\ & \frac{\mathrm{d}v'}{\mathrm{d}x} = -\frac{2}{x^3+x}v' \\
\Longrightarrow\ & \int \frac{1}{v'}\mathrm{d}v' = -2\int \frac{1}{x^3+x}\mathrm{d}x = -2\int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)\mathrm{d}x \\
\Longrightarrow\ & \ln v' = -2\ln(x) + \ln(x^2+1) + C \qquad [C \in \mathbb{R}] \\
\Longrightarrow\ & v' = \mathrm{e}^C\left(\frac{x^2+1}{x^2}\right) = \mathrm{e}^C\left(1 + \frac{1}{x^2}\right) \\
\Longrightarrow\ & v = \mathrm{e}^C\left(x - \frac{1}{x}\right).
\end{aligned}$$

So,

$$(35) \qquad y_2 = xv = \mathrm{e}^C x\left(x - \frac{1}{x}\right) = \mathrm{e}^C(x^2 - 1).$$

Let $B = \mathrm{e}^C$ and let $A \in \mathbb{R}$ be an arbitrary constant. Since $y_2 = B(x^2-1)$ is a solution and $y_1 = x$ is a solution (hence $y_1 = Bx$ is a solution), $y = y_1 + y_2$ is the general solution. So, $y = A(x^2-1) + Bx$ is the solution to the differential equation. Wolfram Alpha confirms this.

Series method:
Assume a solution as in (6), so that (7) follows. Then we plug (6) and (7) into the differential equation to get

$$(36) \qquad (x^2+1)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=0}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Thus,

$$(37) \qquad \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} (n(n-1) - 2n + 2)a_n x^n = 0.$$

So,

$$(38) \qquad \sum_{n=0}^{\infty} (n(n-1) - 2n + 2)a_n x^n = -\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n,$$

and a_0 and a_1 are arbitrary constants. Therefore,

$$(39) \qquad (n(n-1) - 2n + 2)a_n = -a_{n+2}(n+2)(n+1),$$

so

$$(40) \qquad a_{n+2} = -\frac{n(n-1) - 2n + 2}{(n+2)(n+1)}a_n = -\frac{n^2 - 3n + 2}{n^2 + 3n + 2}a_n.$$

Then $a_2 = -a_0$, $a_3 = -\dfrac{1-3+2}{1+2+3}a_1 = 0$, $a_4 = -\dfrac{4-6+2}{4+6+2}a_2 = 0$. Thus, for $n \geq 2$ we have $a_n = 0$. So, the solution is

$$y = \sum_{n=0}^{2} a_n x^n = a_0 + a_1 x + -a_2 x^2 = a_0\left(1 - x^2\right) + a_1 x. \tag{41}$$

If we set $A = -a_0$ and $B = a_1$ then we get the same solution we found earlier, namely

$$y = A\left(x^2 - 1\right) + Bx. \tag{42}$$

PROBLEMS, SECTION 12.2

2.
Show that $P_l(-1) = (-1)^l$.

The Legendre differential equation is

$$\left(1 - x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0. \tag{43}$$

A general solution of (43) is

$$\begin{aligned} y(x) = a_0&\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \cdots\right] \\ &+ a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{4!}x^5 - \cdots\right]. \end{aligned} \tag{44}$$

Put $-x$ in place of x in the above equation. Then we get

$$\begin{aligned} y(-x) = a_0&\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \cdots\right] \\ &- a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{4!}x^5 - \cdots\right]. \end{aligned} \tag{45}$$

The Legendre polynomials $P_l(x)$ are just special cases of (44) where l is a positive integer and we have the restriction $P_l(1) = 1$. For odd l we choose $a_0 = 0$ and for even l we choose $a_1 = 0$. Thus, for odd l we have

$$P_l(x) = a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{4!}x^5 - \cdots\right], \tag{46}$$

so

$$\begin{aligned} P_l(-x) &= a_1\left[-x - \frac{(l-1)(l+2)}{3!}(-x)^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{4!}(-x)^5 - \cdots\right] \\ &= -a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{4!}x^5 - \cdots\right] \\ &= -P_l(x) = (-1)^l P_l(x). \end{aligned} \tag{47}$$

Similarly, for even l we have

$$P_l(x) = a_0\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \cdots\right], \tag{48}$$

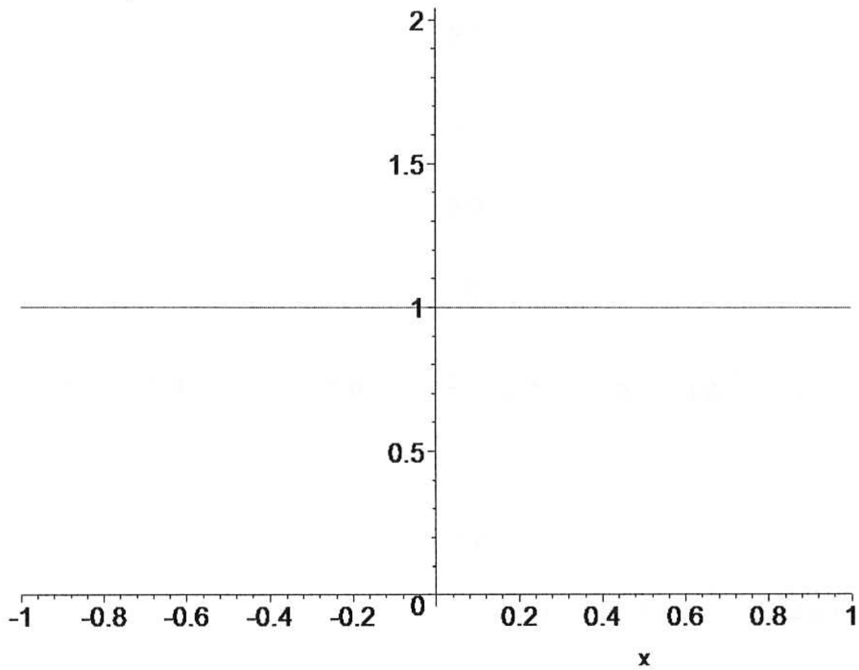
so

$$P_l(-x) = a_0\left[1 - \frac{l(l+1)}{2!}(-x)^2 + \frac{l(l+1)(l-2)(l+3)}{4!}(-x)^4 - \cdots\right] \tag{49}$$
$$= a_0\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \cdots\right]$$
$$= P_l(x) = (-1)^l P_l(x).$$

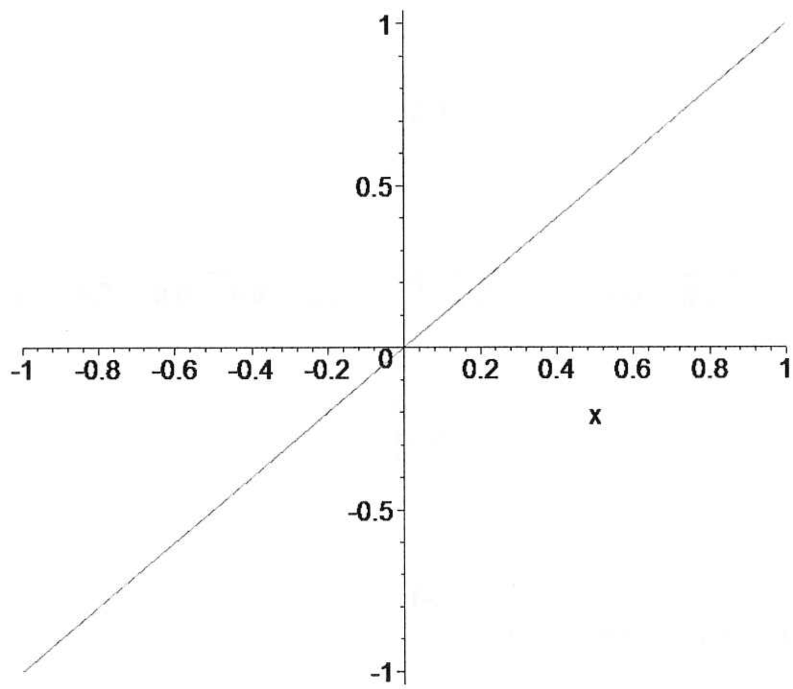
Thus, for any $l \in \mathbb{N}$, we have $P_l(-x) = (-1)^l P_l(x)$. Since we construct P_l such that $P_l(1) = 1$, it follows that $P(-1) = (-1)^l$, which was to be shown. $\square$

Graphs of Legendre Polynomials

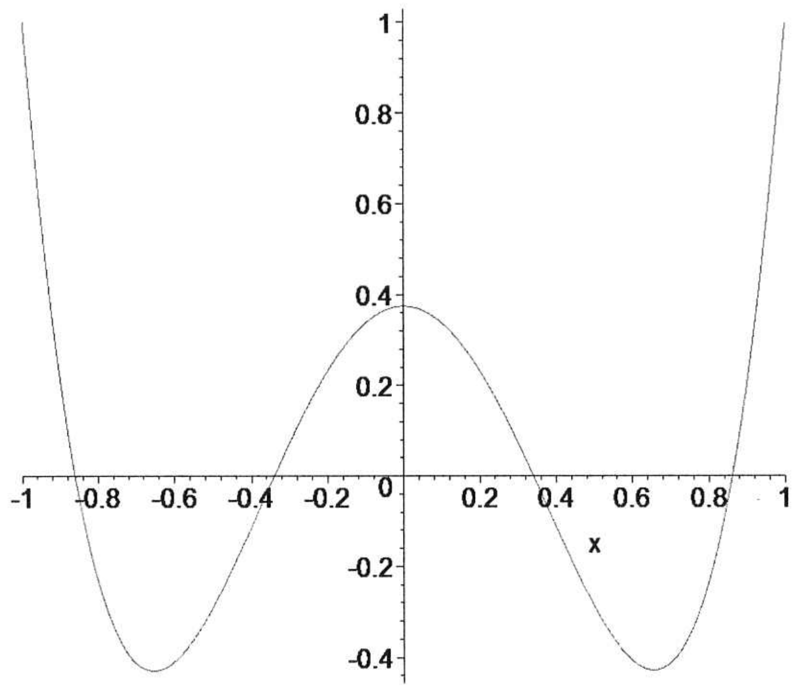
```
> plot(LegendreP(0,x),x=-1..1);
```




```
> plot(LegendreP(1,x),x=-1..1);
```




```
> plot(LegendreP(2,x),x=-1..1);
```


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
x

[>

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Math 342B Assignment 6

Problems, Section 12.5

4.
Show from (5.1) that
$$(x-h)\frac{\partial\Phi}{\partial x} = h\frac{\partial\Phi}{\partial h}.$$
Substitute the series (5.2) for Φ, and so prove the recursion relation (5.8b).

In the textbook, equation (5.1) reads,

$$\Phi(x,h) = \left(1-2xh+h^2\right)^{-1/2}, \qquad |h|<1, \tag{1}$$

and Φ is the generating function for Legendre polynomials. We take partial derivatives of Φ:

$$\begin{aligned}\frac{\partial\Phi}{\partial x} &= -\frac{1}{2}\left(1-2xh+h^2\right)^{-3/2}(-2h) \\ &= h\left(1-2xh+h^2\right)^{-3/2}, \quad \text{and}\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial\Phi}{\partial h} &= -\frac{1}{2}\left(1-2xh+h^2\right)^{-3/2}(-2x+2h) \\ &= (x-h)\left(1-2xh+h^2\right)^{-3/2}.\end{aligned} \tag{3}$$

Multiplying (2) by $x-h$ and (3) by h, we get

$$(x-h)\frac{\partial\Phi}{\partial x} = h(x-h)\left(1-2xh+h^2\right)^{-3/2}, \quad \text{and} \tag{4}$$

$$h\frac{\partial\Phi}{\partial h} = h(x-h)\left(1-2xh+h^2\right)^{-3/2}, \tag{5}$$

so that

$$\boxed{(x-h)\frac{\partial\Phi}{\partial x} = h\frac{\partial\Phi}{\partial h},} \tag{6}$$

as desired. Furthermore, equation (5.2) in the textbook reads

$$\Phi(x,h) = \sum_{l=0}^{\infty} h^l P_l(x). \tag{7}$$

Substituting (7) into (6), we get

$$(x-h)\frac{\partial}{\partial x}\sum_{l=0}^{\infty} h^l P_l(x) = h\frac{\partial}{\partial h}\sum_{l=0}^{\infty} h^l P_l(x). \tag{8}$$

The left hand side of (8) becomes

$$(9) \qquad \begin{aligned}(x-h)\frac{\partial}{\partial x}\sum_{l=0}^{\infty} h^l P_l(x) &= (x-h)\left[\frac{\partial}{\partial x}P_0(x) + \frac{\partial}{\partial x}\sum_{l=1}^{\infty} h^l P_l(x)\right] \\ &= (x-h)\left[\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial x}\sum_{l=1}^{\infty} h^l P_l(x)\right] \\ &= (x-h)\sum_{l=1}^{\infty} h^l P_l'(x) \\ &= \sum_{l=1}^{\infty} h^l x P_l'(x) - \sum_{l=1}^{\infty} h^{l+1} P_l'(x) \\ &= \sum_{l=1}^{\infty} h^l x P_l'(x) - \sum_{l=1}^{\infty} h^l P_{l-1}'(x) \\ &= \sum_{l=1}^{\infty} h^l \big[x P_l'(x) - P_{l-1}'(x)\big].\end{aligned}$$

The second to last step is possible since at $l = 1$, $P_{l-1}'(x) = \frac{\mathrm{d}}{\mathrm{d}x}1 = 0$, so

$$(10) \qquad \begin{aligned}\sum_{l=1}^{\infty} h^{l+1} P_l'(x) &= \sum_{l=1}^{\infty} h^{l+1} P_l'(x) + 0 \\ &= \sum_{l=1}^{\infty} h^{l+1} P_l'(x) + h^1 P_0'(x) \\ &= \sum_{l=0}^{\infty} h^{l+1} P_l'(x) \\ &= \sum_{l=1}^{\infty} h^l P_{l-1}'(x).\end{aligned}$$

Next, the right hand side of (8) becomes

$$(11) \qquad h\frac{\partial}{\partial h}\sum_{l=0}^{\infty} h^l P_l(x) = \sum_{l=0}^{\infty} l h^l P_l(x).$$

Since (9) and (11) are equal, we have

$$(12) \qquad \sum_{l=1}^{\infty} h^l \big[x P_l'(x) - P_{l-1}'(x)\big] = \sum_{l=0}^{\infty} l h^l P_l(x).$$

Thus, for each $l \in N$, the summands must be equal. Therefore, we have

$$(13) \qquad h^l \big[x P_l'(x) - P_{l-1}'(x)\big] = l h^l P_l(x).$$

Dividing both sides of (13) by h^l, we get

$$(14) \qquad \boxed{x P_l'(x) - P_{l-1}'(x) = l P_l(x),}$$

the desired recursion relation.

5.

Differentiate the recursion relation (5.8a) and use the recursion relation (5.8b) with l replaced by $l-1$ to prove the recursion relation (5.8c).

In the textbook, equation (5.8a) is the recursion relation

(15) $$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x).$$

Differentiating (15) with respect to x gives

(16) $$\begin{aligned} lP_l'(x) &= (2l-1)P_{l-1}(x) + (2l-1)xP_{l-1}'(x) - (l-1)P_{l-2}'(x) \\ &= 2lP_{l-1}(x) - P_{l-1}(x) + 2lxP_{l-1}'(x) - xP_{l-1}'(x) - lP_{l-2}'(x) + P_{l-2}'(x). \end{aligned}$$

Recursion relation (5.8b) in the textbook is (once again)

(17) $$xP_l'(x) - P_{l-1}'(x) = lP_l(x).$$

If we make the substitution $l \to l-1$, (17) becomes

(18) $$xP_{l-1}'(x) - P_{l-2}'(x) = (l-1)P_{l-1}(x),$$

so that

(19) $$\begin{aligned} xP_{l-1}'(x) &= (l-1)P_{l-1}(x) + P_{l-2}'(x) \\ &= lP_{l-1}(x) - P_{l-1}(x) + P_{l-2}'(x). \end{aligned}$$

Using (19) in (16) gives

(20) $$\begin{aligned} lP_l'(x) &= 2lP_{l-1}(x) - P_{l-1}(x) + 2l^2P_{l-1}(x) - 2lP_{l-1}(x) + 2lP_{l-2}'(x) \\ &\quad - lP_{l-1}(x) + P_{l-1}(x) - P_{l-2}'(x) - lP_{l-2}'(x) + P_{l-2}'(x) \\ &= \big(2l - 1 + 2l^2 - 2l - l + 1\big)P_{l-1}(x) + (2l - 1 - l + 1)P_{l-2}'(x) \\ &= \big(2l^2 - l\big)P_{l-1}(x) + lP_{l-2}'(x). \end{aligned}$$

Dividing by l on both sides of (20) gives

(21) $$P_l'(x) = (2l-1)P_{l-1}(x) + P_{l-2}'(x).$$

From (19) and (21) we get

(22) $$\begin{aligned} P_l'(x) - xP_{l-1}'(x) &= (2l-1)P_{l-1}(x) + P_{l-2}'(x) - lP_{l-1}(x) + P_{l-1}(x) - P_{l-2}'(x) \\ &= (2l - 1 - l + 1)P_{l-1}(x) + (1-1)P_{l-2}'(x) \\ &= lP_{l-1}(x). \end{aligned}$$

Thus, we get the desired recursion relation:

(23) $$\boxed{P_l'(x) - xP_{l-1}'(x) = lP_{l-1}(x).}$$

12.
Express the polynomial $7x^4 - 3x + 1$ as as a linear combination of Legendre polynomials. *Hint:* Start with the highest power of x and work down in finding the correct combination.

The first five Legendre polynomials are

$$P_0(x) = 1, \tag{24}$$

$$P_1(x) = x, \tag{25}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \tag{26}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad \text{and} \tag{27}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3). \tag{28}$$

Consider the field $\mathbb{R}[x]$ of polynomials in x with real coefficients and the ideal $\langle x^5 \rangle \subseteq \mathbb{R}[x]$. Then the quotient field $\mathbb{R}[x]/\langle x^5 \rangle$ consists of all cosets $p(x) + \langle x^5 \rangle$ with $p(x) \in \mathbb{R}[x]$. This can be thought of as a vector space over $\mathbb{R}/\langle x^5 \rangle$ since $\mathbb{R}/\langle x^5 \rangle$ is a subfield of $\mathbb{R}[x]/\langle x^5 \rangle$ (and since any field is a vector space over one of its subfields). Specifically, if we abandon the formalism of cosets, $\mathbb{R}[x]/\langle x^5 \rangle$ becomes a vector space (call it V) over $\mathbb{R}$ whose vectors are polynomials in x with degree $n \leq 4$. The usual standard basis of V is

$$B = \{1, x, x^2, x^3, x^4\}, \tag{29}$$

So that any polynomial $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 \in V$ can be written as a coordinate vector relative to B:

$$[p(x)]_B = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}^{\mathrm{T}}. \tag{30}$$

Following from (24), (25), (26), (27), and (28), the coordinate vector representations of the Legendre polynomials of degree 4 or less are:

$$[P_0(x)]_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{31}$$

$$[P_1(x)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{32}$$

$$[P_2(x)]_B = \begin{bmatrix} -1/2 & 0 & 3/2 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{33}$$

$$[P_3(x)]_B = \begin{bmatrix} 0 & -3/2 & 0 & 5/2 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \text{and} \tag{34}$$

$$[P_4(x)]_B = \begin{bmatrix} 3/8 & 0 & -15/4 & 0 & 35/8 \end{bmatrix}^{\mathrm{T}}. \tag{35}$$

Let $p(x) = 7x^4 - 3x + 1$, the polynomial we are trying to express as a linear combination of Legendre polynomials. The coordinate vector of $p(x)$ is

$$[p(x)]_B = \begin{bmatrix} 1 & -3 & 0 & 0 & 7 \end{bmatrix}^{\mathrm{T}}. \tag{36}$$

Our problem now becomes finding coefficients $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that

$$p(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + a_4 P_4(x). \tag{37}$$

Thus, we get the following matrix equation that is equivalent to (37):

$$\begin{bmatrix} 1 & 0 & -1/2 & 0 & 3/8 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 3/2 & 0 & -15/4 \\ 0 & 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 0 & 35/8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 7 \end{bmatrix}. \tag{38}$$

For the 5×5 sparse upper triangular matrix in (38), it is easy to find its inverse using elementary row reduction:

$$\begin{bmatrix} 1 & 0 & -1/2 & 0 & 3/8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3/2 & 0 & -15/4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5/2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 35/8 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

$$\to \begin{bmatrix} 1 & 0 & -1/2 & 0 & 3/8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5/2 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 8/35 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 & 1 & 0 & 0 & 0 & -3/35 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2/3 & 0 & 4/7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 8/35 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1/3 & 0 & 7/35 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2/3 & 0 & 4/7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 8/35 \end{bmatrix}$$

Thus, the solution to the system in (38) is

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/3 & 0 & 7/35 \\ 0 & 1 & 0 & 3/5 & 0 \\ 0 & 0 & 2/3 & 0 & 4/7 \\ 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 8/35 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 12/5 \\ -3 \\ 4 \\ 0 \\ 8/5 \end{bmatrix}. \tag{40}$$

Therefore, the Legendre polynomial expansion of $p(x) = 7x^4 - 3x + 1$ is

$$\boxed{7x^4 - 3x + 1 = \frac{12}{5}P_0(x) - 3P_1(x) + 4P_2(x) + \frac{8}{5}P_4(x).} \tag{41}$$