

10.6 Alternating Series and Conditional Convergence

The Alternating Series Test :

The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$

converges if all three of the following conditions are satisfied:

- 1- The u_n 's are all positive
- 2- the positive u_n 's are (eventually) nonincreasing
 $u_n \geq u_{n+1}$ for all $n \geq N$ for some integer N
- 3- $\lim_{n \rightarrow \infty} u_n = 0$

Def. - A convergent series that is not absolutely convergent is conditionally convergent.

Questions: 8, 13, 20, 25, 30, 39, 42, 50, 54

[8] Determine if the alternating series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$$

n^{th} term test: $\lim_{n \rightarrow \infty} (-1)^{n+1} \cdot \frac{n^2+5}{n^2+4} = \text{DNE}$

(Note that $\lim_{n \rightarrow \infty} \frac{n^2+5}{n^2+4} = 1$)
so the series diverges.

[13] Determine if the alternating series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+1}}{n+1}$$

use the alternating series test:-

$$f(x) = \frac{\sqrt{x}+1}{x+1}$$

$$f'(x) = \frac{(x+1) \cdot \frac{1}{2\sqrt{x}} - (\sqrt{x}+1) \cdot 1}{(x+1)^2} = \frac{x+1 - 2x - 2\sqrt{x}}{2\sqrt{x}(x+1)^2}$$

$$= \frac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2} < 0 \text{ for } x \geq 1$$

$f(x)$ is decreasing

$$u_n \geq u_{n+1} \quad \text{for } n \geq 1$$

$$u_n = \frac{\sqrt{n} + 1}{n + 1} \geq 0 \quad \text{for } n \geq 1$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 1}{n + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2\sqrt{n}}}{1} = \frac{0}{1} = 0$$

So the series ~~diverges~~ converges by A.S.T.

$$\boxed{20} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$$

use the n th term test.

$$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{2 \cdot 2 \cdot 2 \cdot 2 \cdots 2} \geq \lim_{n \rightarrow \infty} \frac{n}{2} = \infty$$

$$\text{So } \lim_{n \rightarrow \infty} \frac{n!}{2^n} = \infty$$

So the series diverges.

25 Determine if the series converges absolutely, converges or diverges?

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$$

- Converges absolutely?? $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1+n}{n^2} \right|$ Converges??

$$\sum_{n=1}^{\infty} \frac{1+n}{n^2} \text{ Converges??}$$

$$\sum_{n=1}^{\infty} \frac{1+n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n} = \text{diverges.}$$

$\downarrow$ Converges $\downarrow$ diverges

So the series doesn't converge absolutely.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2} \text{ Converges??}$$

by A.S.T :- ① $u_n = \frac{1+n}{n^2} \geq 0$ for $n \geq 1$

$$\textcircled{2} \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1+n}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{2n} = 0$$

$$\textcircled{3} f(x) = \frac{1+x}{x^2}, \quad f'(x) = \frac{x^2 \cdot 1 - (1+x) \cdot 2x}{(x^2)^2} = \frac{x^2 - 2x - 2x^2}{x^4}$$

$$f'(x) = \frac{-x^2 - 2x}{x^4} < 0 \text{ for } x \geq 1$$

So by the A.S.T, the series converges.

So the series converges conditionally.

Which of the following converge absolutely, which converge, and which diverge??

$$\boxed{30} \quad \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$$

- converge absolutely ?? $\sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$??

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic series)

$$\lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n - \ln n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n \ln n}{n - \ln n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{\ln n}{n}}{1 - \frac{1}{n}} = \frac{\infty}{1} = \infty$$

So the series $\sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$ diverges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n} \quad ??$$

Let ① $u_n = \frac{\ln n}{n - \ln n} \geq 0$ for $n \geq 1$ (Note that $n > \ln n$)

$$\textcircled{2} \quad \lim_{n \rightarrow \infty} \frac{\ln n}{n - \ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 - \frac{1}{n}} = \frac{0}{1-0} = 0$$

$$\textcircled{3} \quad f(x) = \frac{\ln x}{x - \ln x}, \quad f'(x) = \frac{(x - \ln x) \cdot \frac{1}{x} - \ln x (1 - \frac{1}{x})}{(x - \ln x)^2}$$

$$f'(x) = \frac{1 - \frac{\ln x}{x} - \ln x + \frac{\ln x}{x}}{(x - \ln x)^2} = \frac{1 - \ln x}{(x - \ln x)^2}$$

for $x \geq e$, $\ln x \geq 1 \rightarrow 1 - \ln x \leq 0 \rightarrow f'(x) \leq 0$

So the series converges by A.S.T

So the series converges conditionally.

$$\boxed{39} \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{2^n n! n}$$

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n! n} \text{ converges??}$$

$$\begin{aligned} n^{\text{th}} \text{ term test: } \lim_{n \rightarrow \infty} \frac{(2n)!}{2^n n! n} &= \lim_{n \rightarrow \infty} \frac{2n(2n-1)(2n-2) \dots n!}{2^n n! n} \\ &= \lim_{n \rightarrow \infty} \frac{2n(2n-1)(2n-2) \dots (n+1)}{2^{n-1} n!} = \lim_{n \rightarrow \infty} \frac{(2n+1)(2n-2) \dots (n+1)}{2^{n-1}} \end{aligned}$$

$$> \lim_{n \rightarrow \infty} \frac{(n+1)^{n-1}}{2^{n-1}} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{2} \right)^{n-1} = \infty$$

$$\text{So } \sum_{n=1}^{\infty} \frac{(2n)!}{2^n n! n} \text{ diverges}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{2^n n! n}$$

$$n^{\text{th}} \text{ term test: } \lim_{n \rightarrow \infty} \frac{(-1)^n (2n)!}{2^n n! n} = \text{DNE}$$

$$\text{So } \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{2^n n! n} \text{ diverges}$$

$$[42] \quad \sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2+n} - n)$$

$$\lim_{n \rightarrow \infty} (\sqrt{n^2+n} - n) = \lim_{n \rightarrow \infty} (\sqrt{n^2+n} - n) \cdot \left(\frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{n^2+n - n^2}{\sqrt{n^2+n} + n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2(1+\frac{1}{n})} + n} = \lim_{n \rightarrow \infty} \frac{n}{n(\sqrt{1+\frac{1}{n}} + 1)} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

So by the n^{th} term test, $\sum_{n=1}^{\infty} (\sqrt{n^2+n} - n)$ diverges

and by the n^{th} term test, $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2+n} - n)$

diverges since $\lim_{n \rightarrow \infty} (-1)^n (\sqrt{n^2+n} - n) = \text{DNE}$

50 Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$|E| < |a_5|$$

$$|E| < |(-1)^{5+1} \frac{1}{10^5}|$$

$$|E| < \left| \frac{1}{10^5} \right| = 0.00001$$

54 Determine how many terms should be used to estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ with an error of less than 0.001

$$|u_{n+1}| < 0.001 \rightarrow \frac{n+1}{(n+1)^2+1} < \frac{1}{1000}$$

$$1000(n+1) < (n+1)^2+1$$

$$1000n + 1000 < n^2 + 2n + 1 + 1$$

$$0 < n^2 - 998n - 998$$

$$n > \frac{998 + \sqrt{(998)^2 - 4(1)(-998)}}{2} \approx 998.99$$

$$n \geq 999$$