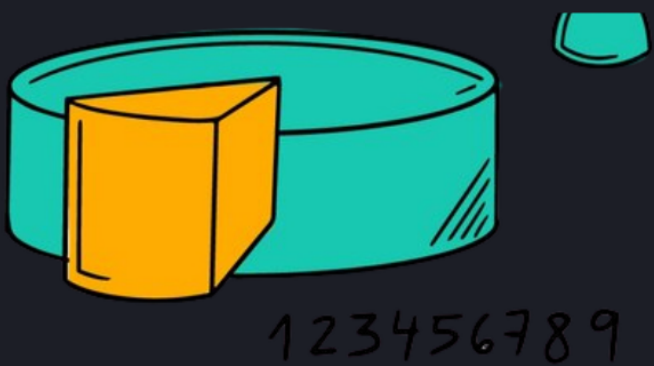
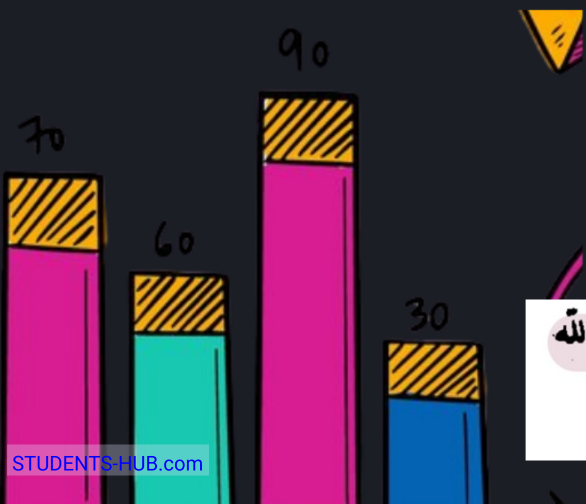
STATISTICS

By Rawan Alfares





ومن أيقن انه يتبع رسولاً من أولي العزم ، صلى الله عليه وسلم ، فكيف لا يستمد من عزمه ؟

8-1-2024

EXAMPLE (4-2):

Suppose that the random variable (X) can take only the values (1 , 2 , 3) and the random variable (Y) can take only the values (1 , 2 , 3 , 4). The joint pdf is shown in the table.

X \ Y	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

$\to P(X=1, Y=1)$

$\to P(X=1, Y=2)$

$\to P(X=1, Y=3)$

1- Find $f_X(x)$ and $f_Y(y)$.

2- Find $P(X \geq 2)$

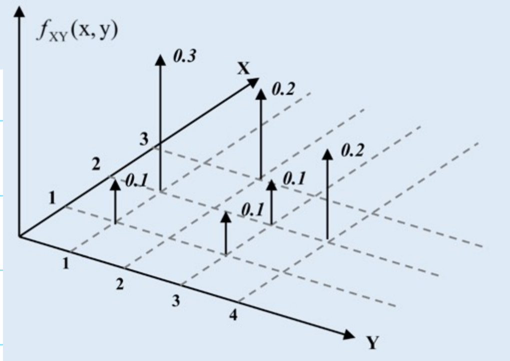
3- Are (X) and (Y) independent.

4- Find the mean and variance for both X and Y

5- Find $P(X > Y)$

6- Find $P(X = Y)$

7- Find the correlation coefficient between X and Y.



$PMF_X = P(X=x)$ } in Single Random varible

$PMF_Y = P(Y=y)$

joint $PMF_{X,Y} = P(X=x, Y=y)$

probility mass function Can be described in 3 ways :-

1. using Schedual.

2. $P(X=x, Y=y) = \begin{cases} 0.1, & x=1, y=1 \\ 0, & x=1, y=2 \\ 0 & x=1, y=3 \\ & x=1, y=4 \\ \vdots & \end{cases}$

$PMF_{X,Y}$ ⇓ joint

3. graphs , "gonna be 2D"

10-1-2024

* the sum of probability should equals one.

examples :-

1. $P(X=3, Y=2) = 0.2$

2. $P(X=1, Y\leq 3) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$= 0.1 + 0 + 0.1 = 0.2$

3. $P(X\leq 2, Y\leq 2) = 0.3 + 0 + 0.1 + 0 = 0.4$

4. $F_{x,y}(2,2)$ = joint CDF of x and y.

$= p(X \leq 2, Y \leq 2)$

Same as 3.

5. $F_{x,y}(\infty, -\infty) = p(X \leq \infty, Y \leq -\infty)$

$= \text{Zero}.$

6. $F_{x,y}(\infty, \infty) = p(X \leq \infty, Y \leq \infty)$

$= 1$

7. $p(X=Y) = 0.1 + 0 + 0 = 0.1$

8. $p(X > Y) = 0.3 + 0 + 0.2 = 0.5$

9. $p(|X-Y|=1) = 0.3 + 0.1 + 0.2 = 0.6$

10. $p(X=1) = 0.1 + 0 + 0.1 + 0 = 0.2$

11. $p(Y=1) = 0.1 + 0.3 + 0 = 0.4$

* for joint probability mass function, $\sum p(X=x) = 1$

$\sum p(Y=y) = 1$

and $\sum p(X=x, Y=y) = 1$

* $p(X=x) = \begin{cases} 0.2, & x=1 \\ 0.6, & x=2 \\ 0.2, & x=3 \\ 0, & o.w \end{cases}$

$\mu_X = (1)(0.2) + (2)(0.6) + (3)(0.2)$

$= 2$

$p(Y=y) = \begin{cases} 0.4, & y=1 \\ 0.2, & y=2 \\ 0.2, & y=3 \\ 0.2, & y=4 \\ 0, & o.w \end{cases}$

$\mu_Y = (1)(0.4) + (0.2)(2) + (0.2)(3)$

$+ (0.2)(4) = 2.2$

$\sigma_X^2 = E[X^2] - \mu_X^2$

$E[X^2] = (1)(0.2) + (4)(0.6) + (9)(0.2) = 4.4$

$\sigma_X^2 = 4.4 - 4 = 0.2$

$\sigma_Y^2 = E[Y^2] - \mu_Y^2$

$E[Y^2] = (1)(0.4) + (4)(0.2) + (9)(0.2)$

$+ (16)(0.2) = 6.2$

$\sigma_Y^2 = 6.2 - (2.2)^2 = 1.36$

$$* \; P(X=x) = \begin{cases} 0.2 & , x=1 \\ 0.6 & , x=2 \\ 0.2 & , x=3 \\ 0 & , 0.w \end{cases}$$

$$P(Y=y) = \begin{cases} 0.4 & , y=1 \\ 0.2 & , y=2 \\ 0.2 & , y=3 \\ 0.2 & , y=4 \\ 0 & , 0.w \end{cases}$$

Are X and y indep? $\longrightarrow P(A \cap B) = P(A) \cdot P(B)$

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

$$P(X=1, Y=1) \overset{??}{=} P(X=1) \cdot P(Y=1)$$

* you can check any pair you want.

$$0.1 \overset{??}{=} (0.2)(0.4)$$

$$0.1 \neq 0.8$$

So, they are not indep.

$$E[XY] = (1)(1)(0.1) + (1)(3)(0.1) + (2)(1)(0.3) + (2)(4)(0.2) + (3)(2)(0.2)$$

$$= 4.4$$

new Concept

correlation coefficient:

معامل الارتباط بين المتغيرين

$$\rho_{xy} = \frac{E(XY) - \mu_x \mu_y}{\sigma_x \sigma_y} = \frac{4.4 - (2)(2.2)}{\sqrt{0.4}\sqrt{1.36}} = 0$$

they are uncolerated

* if they were dependent, that doesn't mean they are colerated.
However, if they were indep, they must be uncolerated.

13-1-2024

$\rho_{xy} = \dfrac{E(XY) - \mu_x \mu_y}{\sigma_x \sigma_y}$ → Covariance $\triangleq \mu_{xy}$

Correlation Coeff. $\rho = \mp 1$ or $\rho = 0$ or $-1 < \rho_{x_i} < 1$

↳ fully Correlated ↳ un Corrlated

* $E[x] = \sum x \, P(X = x)$

$E[XY] = \sum \sum xy \, P(X = x, Y = y)$.

$E[XY] = E[X] . E[Y]$ → only if they were indep.

if X,Y indep $\rho_{x,y} = \dfrac{\mu_x \mu_y - \mu_x \mu_y}{\sigma_x \sigma_y} = 0$

* So, if the two Random Variables were indep, then they are uncorelated

$\boxed{\text{indep} \Rightarrow \text{uncorrelated}}$

dep $\Rightarrow$ الله أعلم , بدنا نفحص ونشوف

* find $E[X^2 Y] = (1)^2(1)(0.1) + (1)^2(3)(0.1) + (2)^2(1)(0.3) + (2^2)(3)(0.1)$
$+ 2^2(4)(0.2) + (3)^2(2)(0.2) = \sim\sim$

* find $E[(X+1)(Y)] = (1+1)(1)(0.1) + (1+1)(3)(0.1) + (2+1)(1)(0.3) \ldots$

⇓

بنجمع لـ X واحد ثم
نضربها بـ Y و P

* find $(P\, X \geq 2 / Y \leq 2) = P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

معناها تقاطع

$= \dfrac{P(X \geq 2, Y \leq 2)}{P(Y \leq 2)}$

* $P(X \geq 2 / Y \leq 2, X \leq 2) = \dfrac{P(X \geq 2, Y \leq 2, X \leq 2)}{P(Y \leq 2, X \leq 2)}$

$= \dfrac{P(X = 2, Y \leq 2)}{P(Y \leq 2, X \leq 2)} = \dfrac{0.3}{0.3 + 0.1} = \dfrac{3}{4} = 0.75$

* let $Z = X+Y$ find PMF_Z, μ_Z, σ_Z^2

z	$P(Z=z)$
2	0.1
3	0 + 0.3
4	0.1
5	0 + 0.1 + 0.2
6	0.2

$$PMF_Z = P(Z=z) = \begin{cases} 0.1, & z=2 \\ 0.3, & z=3 \\ 0.1, & z=4 \\ 0.3, & z=5 \\ 0.2, & z=6 \\ 0, & o.w \end{cases}$$

$\Rightarrow \mu_Z = E[Z] = (2)(0.1)+(3)(0.3)+(4)(0.1)+(5)(0.3)+(6)(0.2)$

$= 4.2$

$\mu_Z = E[X+Y] = E[X] + E[Y]$

محاضرة التعويض

6/1/2025

$E[x^2+x^3] = E[x^2]+E[x^3]$

$= \sum x_i^2 p(x=x_i) + \sum x_i^3 p(x=x_i) \rightarrow$ discreate

$= \int x^2 p(x=x_i)dx + \int x^3 p(x=x_i)dx \rightarrow$ Continuoes.

* if $y = aX+b$

$E[y] = aE[x]+b \rightsquigarrow \mu_y = a\mu_x + b \rightsquigarrow \sigma_y^2 = a^2\sigma_x^2$

* find $E[xy]$, $E[x^2y]$, $E[(x+1)y]$.

$\hookrightarrow E\{g(x,y)\} = \sum_{x_i}\sum_{y_j} g(x_i,y_j)\,p(x=x_i, y=y_j)$

يعني إذا بدنا نحسب الـ E لفنكشن معين $(g(x,y))$، بنضرب قيمة x وقيمة y بالـprobability عندهم وآخر اشي بنعمل Sum .

* find $E[(x^2+y)+xy] = E[(x^2+y)]+E[xy]$

يعني بنوزّع الـ E على كل فنكشن

- ***Theorem: Addition of Means***

The mean or expected value of a sum of random variables is the sum of the expectations.

$E(x_1 + x_2 + \ldots\ldots + x_n) = E(x_1) + E(x_2) + \ldots\ldots + E(x_n)$

لا يوجد عليه شرط سواء كان indep أو dep.

- ***Theorem: Multiplication of Means***

The expected value of the product of independent r.v equals the product of the expected values.

$E(x_1 x_2 \ldots\ldots X_n) = E(x_1)\,E(x_2)\ldots\ldots E(x_n)$

فقط إذا كان indep.

Theorem:

$Y = a_1X_1 + a_2X_2$

$\mu_y = a_1\mu_{x_1} + a_2\mu_{x_2}$

$\sigma_Y^2 = E[(y-\mu_y)^2] = E[y^2]-\mu_y^2$ اشتقاق القانون من الدوسية ص70

حفظ $\rightarrow$ • $\sigma_Y^2 = a_1^2\sigma_{x_1}^2 + a_2^2\sigma_{x_2}^2 + 2a_1a_2\boxed{\sigma_{x_1}\sigma_{x_2}\rho_{x_1,x_2}}$

equals Covariance

حفظ $\rightarrow$ • $\sigma_Y^2 = a_1^2\sigma_{x_1}^2 + a_2^2\sigma_{x_2}^2 + 2a_1a_2(E[xy] - \mu_x\mu_y)$

notes- if x_1 and x_2 were s. Indep $\xrightarrow{\text{then}}$ $\sigma_Y^2 = a_1^2\sigma_{x_1}^2 + a_2^2\sigma_{x_2}^2$

Example:-

let X be a random Variable with $\mu_x = 1$, $\sigma_x^2 = 4$, Y is another Random Variable with $\mu_y = -1$, $\sigma_y^2 = 9$. R is another R.V s.t $R = 2X - Y$

1. $\mu_R = E[R]$

$$= 2E[x] - E[y] = 2\mu_x - \mu_y = (2)(1) - (-1) = 3$$

2. find the Coveriance μ_{xy} if $\rho_{xy} = 0.5$

$$\mu_{xy} = \sigma_x \sigma_y \rho_{xy} = (\sqrt{4})(\sqrt{9})(0.5) = 3$$

3. find σ_R^2 (var) if $\rho_{xy} = 0.5$

$$\sigma_R^2 = a_x^2 \sigma_x^2 + a_y^2 \sigma_y^2 + 2 a_x a_y \sigma_x \sigma_y \rho_{xy}$$

$$= (4)(4) + (-1)^2(9) + (2)(2)(-1)(\sqrt{4})(\sqrt{9})(0.5)$$

$$= 13$$

4. if x, y are S.Indep, find μ_{xy} (covariance), σ_R^2

$$\text{Covariance} = E[xy] - \mu_x \mu_y = E[x].E[y] - \mu_x \mu_y = 0$$

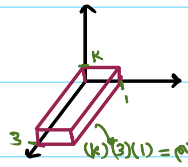
$$\sigma_R^2 = a_x^2 \sigma_x^2 + a_y^2 \sigma_y^2 + 0$$

$$= (4)(4) + (-1)^2(9) = 25$$

Two Continuous Random Variable:

let X and Y be two Random variable with the following joint p.d.f

$$f_{X,Y}(x,y) = \begin{cases} K & , 0 \le x \le 1, 0 \le y \le 3 \\ 0 & , \text{o.w} \end{cases}$$



* الحجم = (k)(3)(1) ، ويجب أن يساوي (1)

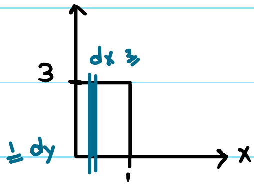
$K = \frac{1}{3}$

a) determine the value of the Constant K.

* one integration gives Area.
* double integration gives volume

[1] method 1.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\, dy\, dx = 1$$



$$\int_0^1\int_0^3 K\, dy dx = 1$$

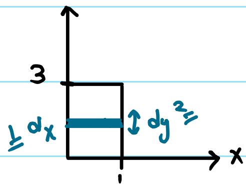
$$\int_0^1 3K\, dx = 1$$

$$3K = 1 \rightarrow \boxed{K = \frac{1}{3}}$$

* أول إشي نكامل التكامل الداخلي ثم الخارجي

[2] method 2.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\, dx\, dy = 1$$



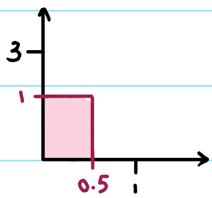
$$\int_0^3\int_0^1 K\, dx\, dy = 1$$

$$\int_0^3 K\, dy = 1$$

$$3K = 1 \rightarrow \boxed{K = \frac{1}{3}}$$

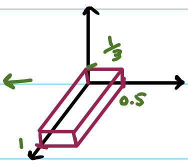
b) $P(0 \le X \le 0.5 , 0 \le Y \le 1)$

$$\int_0^{0.5}\int_0^1 \tfrac{1}{3}\, dy\, dx = \int_0^{0.5} \frac{1}{3} dx = \frac{1}{6}$$



Same answer

volume $= (1)(0.5)(\frac{1}{3})$ ← $= \frac{1}{6}$

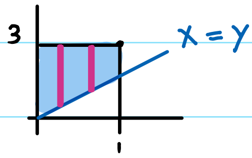


c) $P(X \le Y)$

method 1 :- $\int_0^1\int_x^3 \frac{1}{3}\, dy\, dx = \int_0^1 \frac{1}{3}(3-x)\, dx = \left(x - \frac{x^2}{6}\right)\Big|_0^1 = 1 - \frac{1}{6} = \frac{5}{6}$

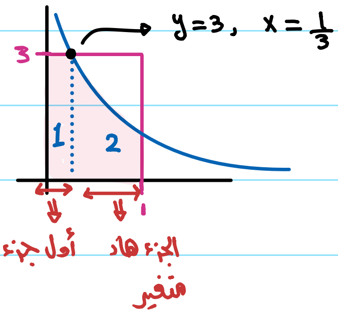
$\frac{9}{6}$

y هي المتغيرة



method 2 :-

d) $p(XY \leq 1)$ $\rightarrow y = \frac{1}{x}$



$$= \int_0^{1/3}\int_0^3 \frac{1}{3}\, dy\, dx + \int_{1/3}^{1}\int_0^{1/x} \frac{1}{3}\, dy\, dx$$

$$= \int_0^{1/3} 1\, dx + \int_{1/3}^{1} \frac{1}{3}\left(\frac{1}{x}\right) dx$$

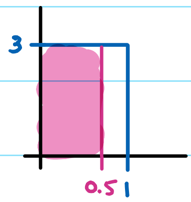
$$= \frac{1}{3} + \frac{1}{3}\ln|x|\Big|_{1/3}^{1} = \frac{1}{3} + \frac{1}{3}\ln(1) - \frac{1}{3}\ln\left(\frac{1}{3}\right) = \frac{1}{3} + \frac{1}{3}\ln(3)$$

e) $p(X \leq 0.5)$ $\rightarrow$ there are two methods to Solve it

① $f_{xy}(X=x, Y=y)$

② $f_x(x)$ $\rightsquigarrow$ margine function

method 1: ① $\int_0^3\int_0^{0.5} \frac{1}{3}\, dx\, dy = \int_0^3 \frac{1}{6}\, dy = \frac{1}{2}$



marginal pdf

$$f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy \qquad f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx$$

method 2: ② $f_x(x) = \int_0^3 \frac{1}{3}\, dy = 1$

$$f_x(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{o.w} \end{cases} \Rightarrow \int_0^{0.5} 1\, dx = 0.5$$

extra: to find $f_y(y)$

$$f_y(y) = \int_0^1 \frac{1}{3}\, dx = \frac{1}{3} \quad , \quad f_y(y) = \begin{cases} \frac{1}{3}, & 0 \leq y \leq 3 \\ 0, & \text{o.w} \end{cases}$$

note that :- $f_x(x) \cdot f_y(y) = f_{xy}(x,y)$ → since they are indep.

$$f_{x,y}(x,y) = \begin{cases} \frac{1}{3} & , 0 \le x \le 1, 0 \le y \le 3 \\ 0 & , o.w \end{cases} = \underbrace{\begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , o.w \end{cases}}_{f_x(x)} \times \underbrace{\begin{cases} \frac{1}{3} & , 0 \le y \le 3 \\ 0 & , o.w \end{cases}}_{f_y(y)}$$

* $PMF_{x,y} = (X = x_i, Y = y_i) = P(X = x_i) \cdot P(Y = y_i)$
* $f_{xy} = f_x(x) \cdot f_y(y)$

Conditional PDF

$$f_{y/x}(y) = \frac{f_{xy}(x,y)}{f_x(x)} \qquad\qquad f_{x/y}(x) = \frac{f_{xy}(x,y)}{f_y(y)}$$

f) $P(y \le 1 / X = 0.5)$

note that x has specific value

So we solve it by using Conditional pdf.

$$f_{y/x}(y) = \left. \frac{1/3}{1} \right|_{X=0.5} = \frac{1}{3} \rightsquigarrow f_{y/x}(y) = \begin{cases} \frac{1}{3} & , 0 \le y \le 3 \\ 0 & , o.w \end{cases}$$

g) $P(0.5 \le X \le 0.75 / Y = 1)$

specific Point (given y)

$$f_{x/y} = \frac{f_{xy}}{f_y(y)} = \left. \frac{1/3}{1/3} \right|_{y=1} = 1 \rightsquigarrow f_{x/y}(x) = \begin{cases} 1 & , 0 \le X \le 1 \\ 0 & , o.w \end{cases}$$

note that $f_{x/y} = f_x(x)$, $f_{y/x} = f_y(y)$

since it's statistically indep.

$$f_{xy}(x,y) = \begin{cases} & , \boxed{0<y<x<2} \\ 0 & , o.w \end{cases}$$

3 intervals

$\rightarrow 0<y<2$

$\rightarrow 0<x<2$

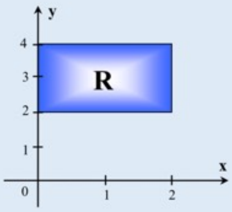
$\rightarrow y<x$

EXAMPLE (4-1):

Let (X) and (Y) be continuous random variables with a joint pdf:

$f_{X,Y}(x,y) = \frac{1}{8}(6-x-y)$; $0 \le x \le 2$, $2 \le y \le 4$

1- Find $f_X(x)$ and $f_Y(y)$.
2- Find the conditional pdf $f_{Y/X}(y)$.
3- Find $P(2 \le y \le 3)$
4- Find $P(2 \le y \le 3 / x=1)$



1) $f_x(x) = \int_2^4 \frac{1}{8}(6-x-y)\,dy = \int_2^4 \frac{(6-x)}{8}dy - \int_2^4 \frac{y}{8}\,dy = \frac{(6-x)}{8}(2) - \frac{y^2}{16}\Big|_2^4$

$= \frac{(6-x)}{4} - \left(\frac{16}{16} - \frac{4}{16}\right)$

$= \frac{(6-x)}{4} - \frac{3}{4} = \frac{(3-x)}{4} = \frac{6-2x}{8}$

$f_y(y) = \int_0^2 \frac{1}{8}(6-y-x)\,dx = \frac{(6-y)(2)}{8} - \frac{x^2}{8}\Big|_0^2 = \frac{(6-y)}{4} - \frac{1}{4} = \frac{5-y}{4}$

---

2) $f_{y/x}(y) = \frac{f_{xy}}{f_x(x)} = \frac{\frac{1}{8}(6-x-y)}{\frac{1}{8}(6-2x)} = \frac{(6-x-y)}{(6-2x)}$

---

3) $P(2 \le y \le 3) = \int_2^3 f_y(y)\,dy = \int_2^3 \frac{5-y}{4} = \frac{5}{4}(1) - \frac{y^2}{8}\Big|_2^3$

$= \frac{5}{4} - \left(\frac{9}{8} - \frac{4}{8}\right) = 0.625$

---

4) $P(2 \le y \le 3 / x=1) \Rightarrow f_{y/x}(y) = \frac{f_{xy}(x,y)}{f_x(x)} = \frac{\frac{1}{8}(6-x-y)}{\frac{1}{8}(6-2x)}\Bigg|_{x=1}$

$= \frac{(5-y)}{4} = f_y(y)$ means its s. indep.

ومن أيقن انه يتبع رسولاً من أولي العزم ، صلى الله
عليه وسلم ، فكيف لا يستمد من عزمه ؟

ربنا تقبل منا إنك
أنت السميع العليم ..

Rawan - Al Fares