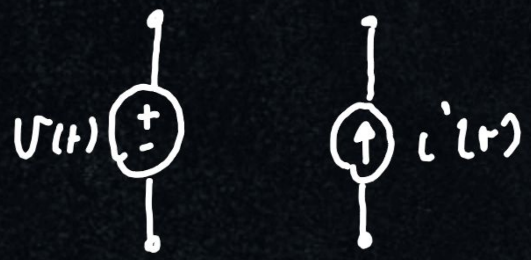
CH 9 : Sinusoidal Steady State Analysis

Sinusoidal source :- The source that produces a voltage or current varies sinusoidally with time.



Voltage source

current source

$$x(t) = X_m \sin(\omega t + \phi) \quad ; \quad x \in \{v, i\}$$

where X_m : peak, magnitude, or amplitude.

ω : Electric radian frequency [rad/sec]

$\omega = 2\pi f$ → Frequency [Hz]

$f = \frac{1}{T}$ → period.

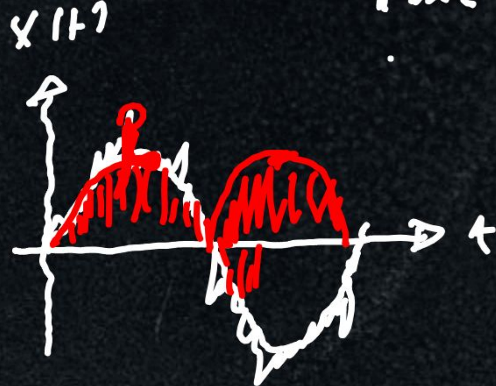
RMS value of x(t)

It is defined as the square root of the mean value of the squared function

$\frac{311}{\sqrt{2}}$

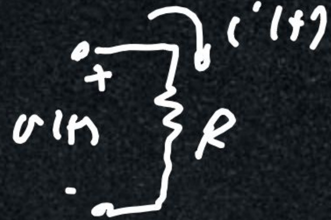
220 V rms

$$X_{RMS} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}$$



$$x(t) = X_m \sin(\omega t + \phi)$$

$$X_{RMS} = \frac{X_m}{\sqrt{2}}$$



$$P = VI$$

$$P = \frac{1}{2} V_m I_m = V_{rms} I_{rms}$$

EX: $v(t) = 300 \cos(\underbrace{120\pi}_{\omega} t + 30^\circ)$

calculate

1) ω

2) f

3) T

4) peak value of $v(t)$

5) RMS value of $v(t)$

6) phase angle of $v(t)$

1) $\omega = 120\pi$ rad/sec

2) $\omega = 2\pi f = 120\pi$

$f = 60$ Hz

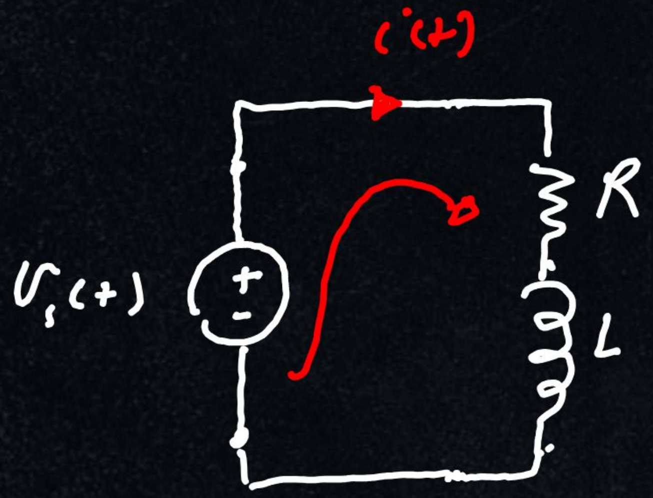
3) $T = 1/60 = 16.67$ msec

4) $V_m = 300$ V

5) $V_{RMS} = \frac{V_m}{\sqrt{2}} = 212.13$ V

6) $\phi = 30^\circ = 30^\circ \frac{\pi}{180}$ rad

Sinusoidal Response



KVL in the loop:-

$$V_s(t) = R\,i(t) + L\frac{di(t)}{dt}$$

$$V_m \cos(\omega t + \phi) = R\,i(t) + L\frac{di(t)}{dt}$$

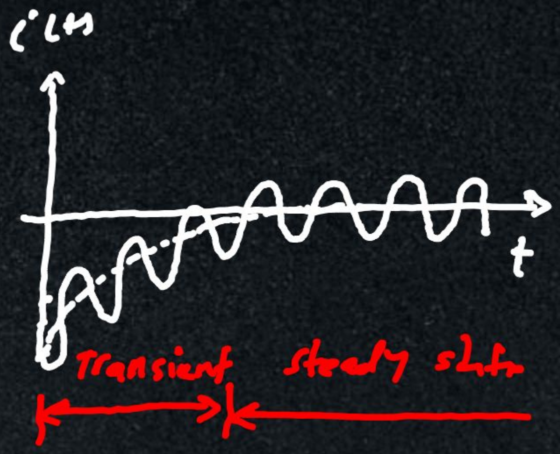
1^{st} order diff. equation
(Linear, non-homog-)

$$V_s(t) = V_m \cos(\omega t + \phi)$$

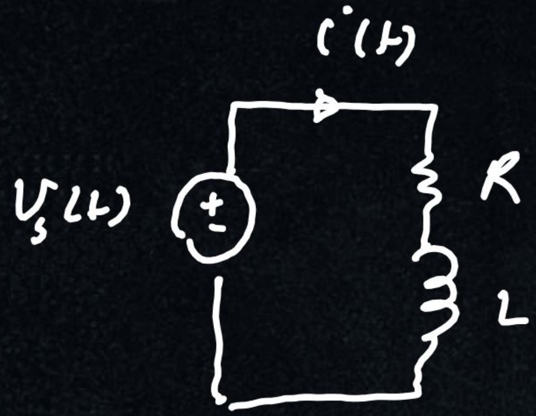
$\tau = L/R$, $Z = \sqrt{(\omega L)^2 + R^2}$
$\theta = \tan^{-1}(\omega L / R)$

The solution is given by:-

$$i(t) = \underbrace{\frac{-V_m}{Z}\cos(\phi - \theta)\, e^{-t/\tau}}_{\text{Transient component}} + \underbrace{\frac{V_m}{Z}\cos(\omega t + \phi - \theta)}_{\text{steady state solution}}$$



Note: In this chapter, we will focous on the steady solution.



source signal $V_s(t)$	Response signal $i_{ss}(t)$
$V_m \cos(\omega t + \phi)$	$\frac{V_m}{Z} \cos(\omega t + \phi - \theta)$

steady state.

characteristics of SS solution

- sinusoidal function
- ω of source signal = ω of response signal
- peak of source signal = V_m ≠ peak of response signal = $\frac{V_m}{Z}$
- phase of source signal = ϕ ≠ phase of response signal = $\phi - \theta$

The phasor

It is a complex number that carries the amplitude and phase angle of sinusoidal function.

$$x(t) = X_m \cos(\omega t + \phi)$$

$$\vec{X} = \mathcal{P}[x(t)] = \mathcal{P}[X_m \cos(\omega t + \phi)] = X_m \angle \phi$$

$$x(t) = \mathcal{P}^{-1}[\vec{X}] = \mathcal{P}^{-1}[X_m \angle \phi] = X_m \cos(\omega t + \phi)$$

Forms of phasor

1) Exponential form

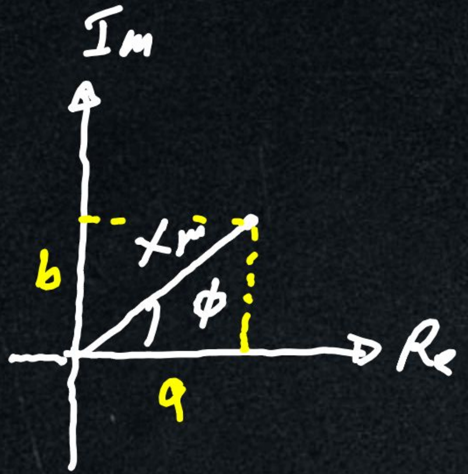
$$\vec{X} = X_m e^{j\phi} \quad ; \quad j = \sqrt{-1}$$

2) Polar form

$$\vec{X} = X_m \angle \phi$$

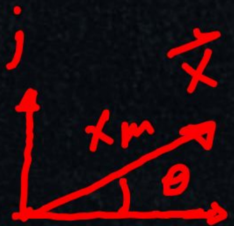
3) Rectangular form

$$\vec{X} = a + jb$$



Polar $\longrightarrow$ Rectangular

$$\vec{X} = X_m \angle\theta$$

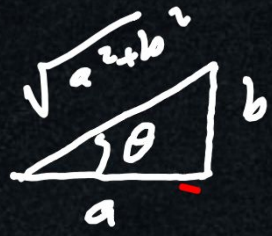


$$\vec{X} = X_m \cos\theta + j X_m \sin\theta$$

Rectangular $\longrightarrow$ Polar

$$\vec{X} = a + jb$$

$$\vec{X} = \left(\sqrt{a^2+b^2}\right) \angle \tan^{-1}(b/a)$$



$\underline{Ex}$:- $\vec{Y}_1 = 20\angle -30^\circ$, $\vec{Y}_2 = 40\angle 60^\circ$

$\underline{Ae^{j\theta_a}}$ $\underline{Be^{j\theta_b}}$

Find 1) $\vec{Y}_1 + \vec{Y}_2$

2) $\vec{Y}\,\vec{Y}_2$

1) $\vec{Y}_1 + \vec{Y}_2 = 20\angle -30^\circ + 40\angle 60^\circ$

$= (7.32 - j10) + (20 + j34.64)$

$= 27.32 + j24.64$

$= 44.72\angle 33.43^\circ$

2) $\vec{Y}_1\,\vec{Y}_2$

$= (20(40))\angle -30^\circ + 60^\circ$

$= 800\angle 30^\circ$

Passive elements in phasor-domain

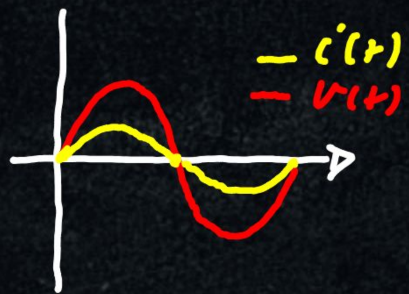
① Resistor

$$v(t) = R\,i(t)$$

$i(t)$ R

$+\; v(t) \;-$

t-domain



Assume that

$$i(t) = I_m \cos(\omega t + \theta_i) \xrightarrow{P} \vec{I} = I_m \angle \theta_i$$

$$v(t) = R I_m \cos(\omega t + \theta_i) \xrightarrow{P} \vec{V} = (R I_m) \angle \theta_i$$

$$\vec{V} = R(I_m \angle \theta_i) = R\,\vec{I}$$

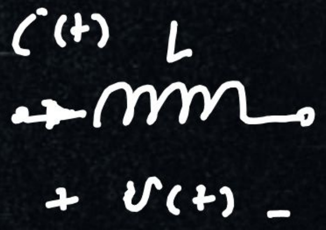
$$\boxed{\vec{V} = R\,\vec{I}}$$

$\vec{I}$ R

$+\; \vec{V} \;-$

The current & voltage are in-phase.

② Inductor



t-domain

$$v = L\frac{di}{dt}$$

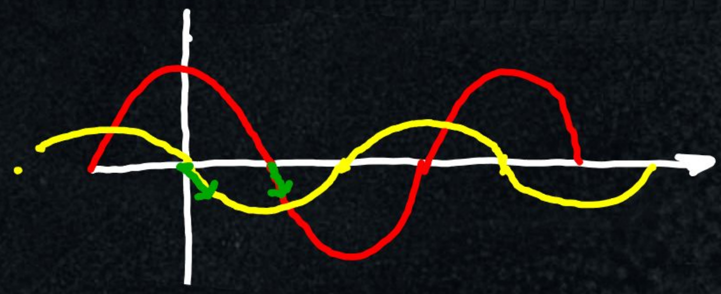
Assume that

$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$\vec{I} = I_m \angle \theta_i$$

$\theta_v = \theta_i + 90°$

$\theta_i = \theta_i$



i lags v by 90°

$\theta_v - \theta_i = 90°$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i) = \omega L I_m \sin(\omega t + \theta_i + 180°)$$

$$v = \omega L I_m \cos(\omega t + \underline{\theta_i + 90°})$$

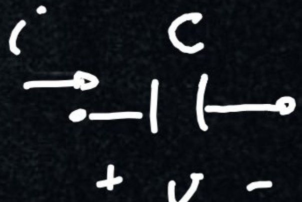
$$\vec{V} = \omega L I_m \angle \theta_i + 90° = \omega L I_m e^{j(\theta_i + 90°)} = \omega L I_m e^{j\theta_i} e^{j90°}$$

$e^{j90°} = j1$

$$\vec{V} = j\omega L \, \underline{I_m e^{j\theta_i}} \Rightarrow \vec{V} = (j\omega L)\vec{I}$$

phasor domain

③ Capacitor



t-domain

$$i = C\frac{dv}{dt}$$

Assume $v = V_m \cos(\omega t + \theta_v)$

$$i = -\omega C V_m \sin(\omega t + \theta_v)$$

$$i = \omega C V_m \sin(\omega t + \theta_v + 180^\circ)$$

$$i = \omega C V_m \cos(\omega t + \theta_v + 90^\circ)$$

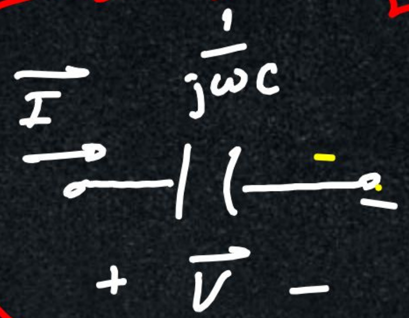
i leads v by 90°

$$\vec{I} = \omega C V_m \angle \theta_v + 90^\circ = \omega C V_m e^{j(90^\circ + \theta_v)}$$

$$\vec{I} = \omega C V_m e^{j\theta_v} e^{j90^\circ}$$

$$\vec{I} = j\omega C \underbrace{V_m e^{j\theta_v}}_{\vec{V}} \Rightarrow \vec{I} = (j\omega C)\vec{V}$$

$$\vec{V} = \left(\frac{1}{j\omega C}\right)\vec{I}$$



phasor domain.

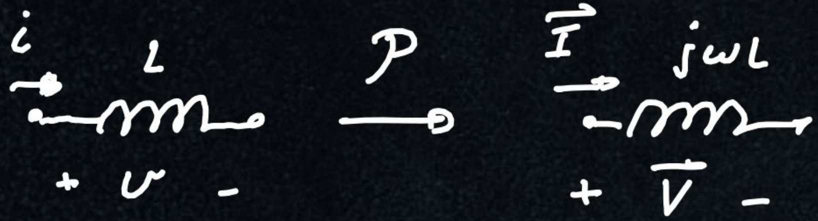
	t-domain	phasor-domain	
Resistor	R	R	i & v are in phase.
Inductor	L	$j\omega L$	i lags v by 90°
Capacitor	C	$\frac{1}{j\omega C} = \frac{-j}{\omega C}$	i leads v by 90°

Passive Elements in phasor-domain

1) 

$v = R\,i$ $\qquad\qquad$ $\vec{V} = R\,\vec{I}$

i & v are in-phase

2) 

$v = L\dfrac{di}{dt}$ $\qquad\qquad$ $\vec{V} = (j\omega L)\,\vec{I}$

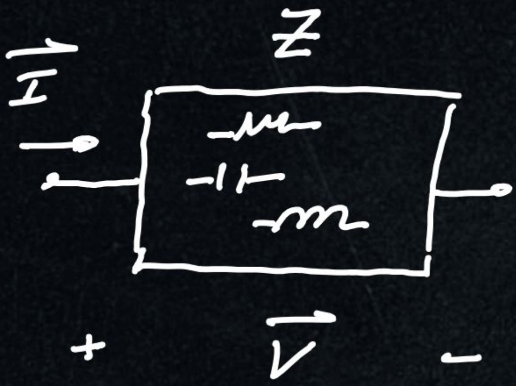
i lags v by 90° $\qquad \theta_v = \theta_i + 90^\circ$

3) 

$i = C\dfrac{dv}{dt}$ $\qquad\qquad$ $\vec{V} = \left(\dfrac{1}{j\omega C}\right)\vec{I}$

i leads v by 90° $\qquad \theta_i = \theta_v + 90^\circ$

Impedance



$$\vec{V} = Z\,\vec{I}$$

impedance

The impedance is measured in $[\Omega]$

$$Z = R + jX$$

impedance $[\Omega]$

resistance. $[\Omega]$

reactance $[\Omega]$

	R	X	Z
R	R	0	R
L	0	ωL	$j\omega L$
C	0	$\frac{-1}{\omega C}$	$\frac{-j}{\omega C}$

$Z = R$

$Z = j\omega L$

$Z = -j/\omega C$

Admittance :- It is the reciprocal of impedance and measured in siemens or [℧]

$$Y = \frac{1}{Z} = \text{complex number}$$

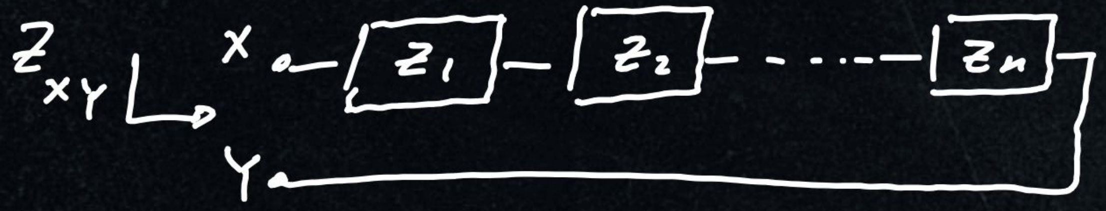
$$Y = G + jB$$

Add. [℧]

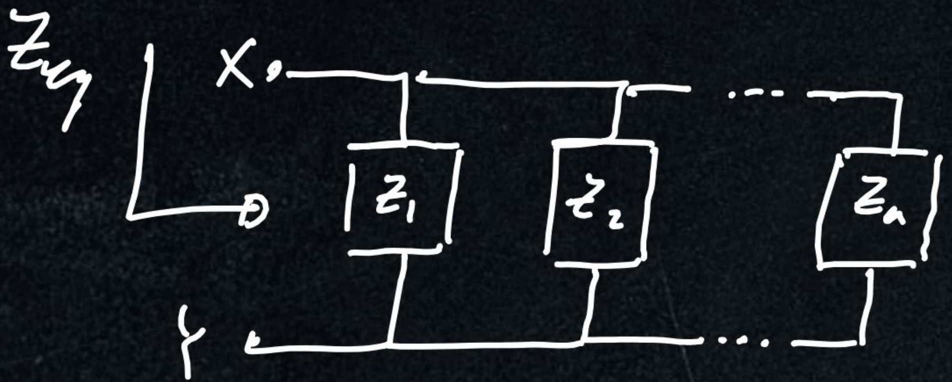
conductance [℧]

Susceptance [℧]

Impedances in series

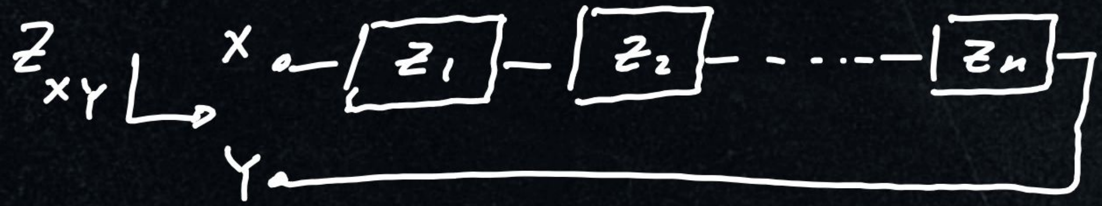


$$Z_{eq} = Z_{XY} = Z_1 + Z_2 + \cdots + Z_n$$

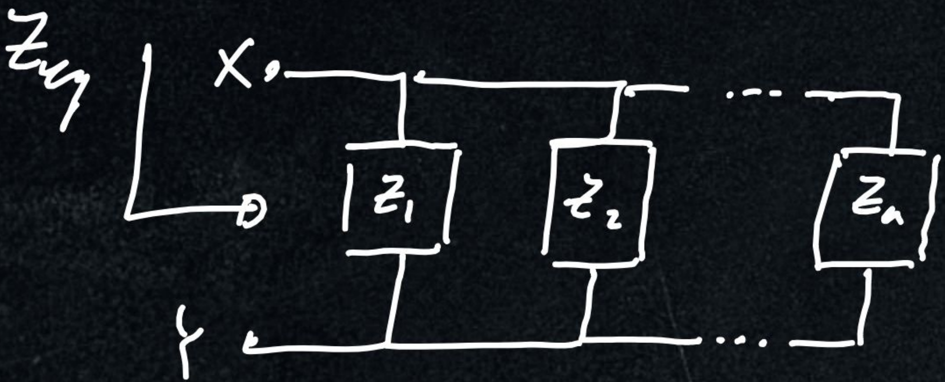


$$Z_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}}$$

Impedances in series



$$Z_{eq} = Z_{XY} = Z_1 + Z_2 + \cdots + Z_n$$



$$Z_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}}$$

KVL & KCL

t-domain

$$\sum i \Big|_{node} = 0$$

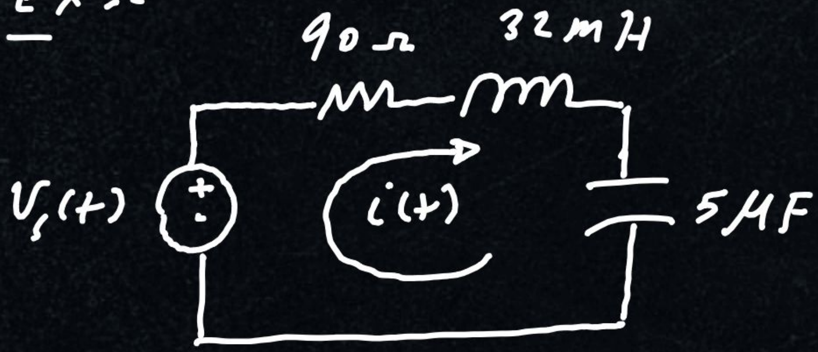
$$\sum v \Big|_{loop} = 0$$

phasor-domain

$$\sum \vec{I} \Big|_{node} = 0$$

$$\sum \vec{V} \Big|_{loop} = 0$$

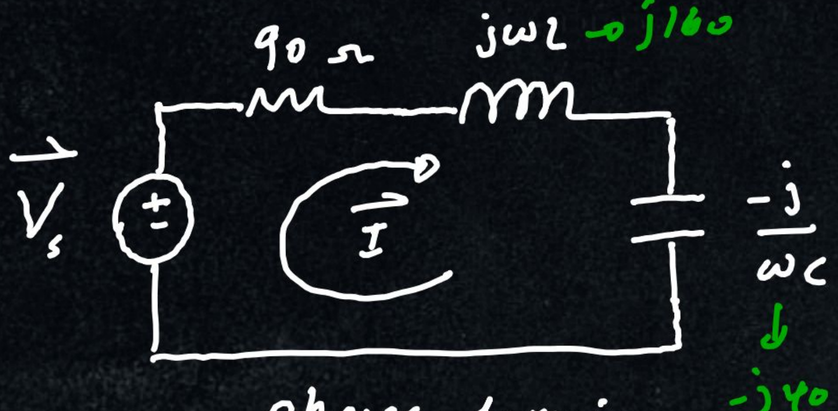
Ex:-



t-domain

calculate the steady state current

$v_s(t) = 750\cos(\underbrace{5000}_{\omega}t + 30^\circ)$ V



phasor-domain

$\vec{V}_s = 750\angle 30^\circ$ V

$j\omega L = j5000 \times 32 \times 10^{-3} = j160\ \Omega$

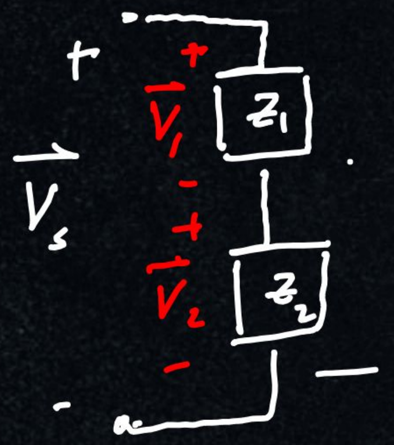
$\frac{-j}{\omega C} = \frac{-j}{5000 \times 5 \times 10^{-6}} = -j40\ \Omega$

$$\vec{I} = \frac{(\vec{V}_s = 750\angle 30^\circ)}{90 + j160 - j40} = \frac{750\angle 30^\circ}{90 + j120} = \frac{750\angle 30^\circ}{150\angle 53.13^\circ} = 5\angle -23.13^\circ$$

$\vec{I} = 5\angle -23.13^\circ$ A $\Rightarrow$ $\boxed{i(t) = 5\cos(5000t - 23.13^\circ)\ A}$

Voltage + Current Dividers

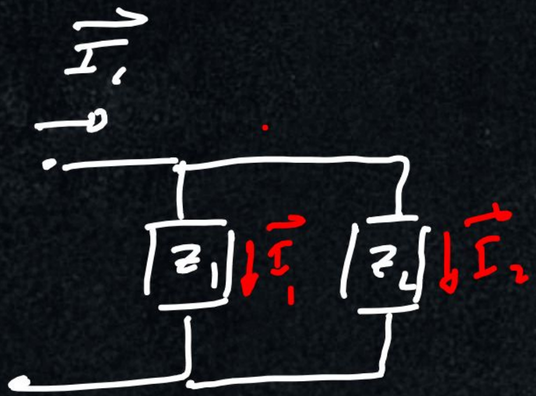
VD



$$\vec{V}_1 = \frac{Z_1}{Z_1 + Z_2} \cdot \vec{V}_s$$

$$\vec{V}_2 = \frac{Z_2}{Z_1 + Z_2} \vec{V}_s$$

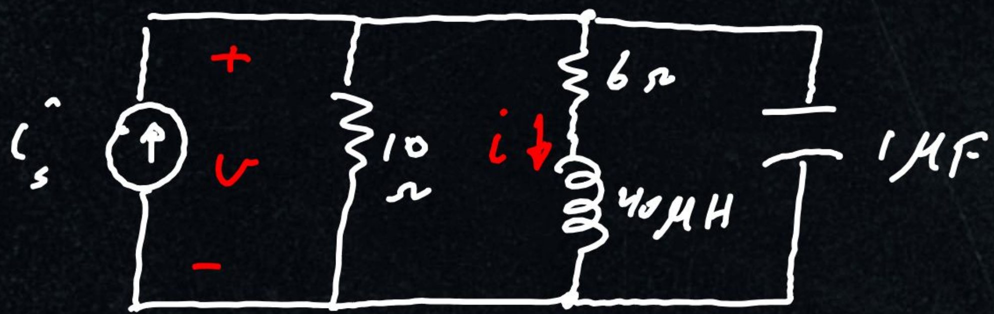
CD



$$\vec{I}_1 = \frac{Z_{eq}}{Z_1} \cdot \vec{I}_s = \frac{Z_2}{Z_1 + Z_2} \vec{I}_s$$

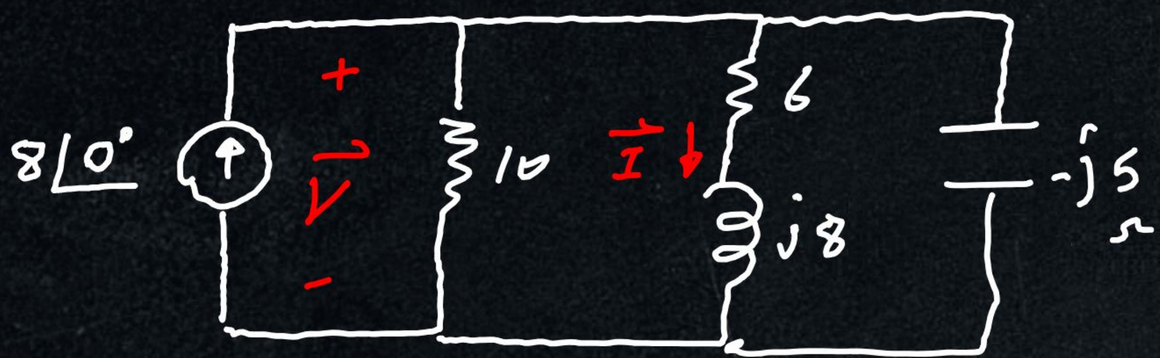
$$\vec{I}_2 = \frac{Z_{eq}}{Z_2} \vec{I}_s = \frac{Z_1}{Z_1 + Z_2} \cdot \vec{I}_s$$

Ex :



Calculate the steady state expressions for v & i

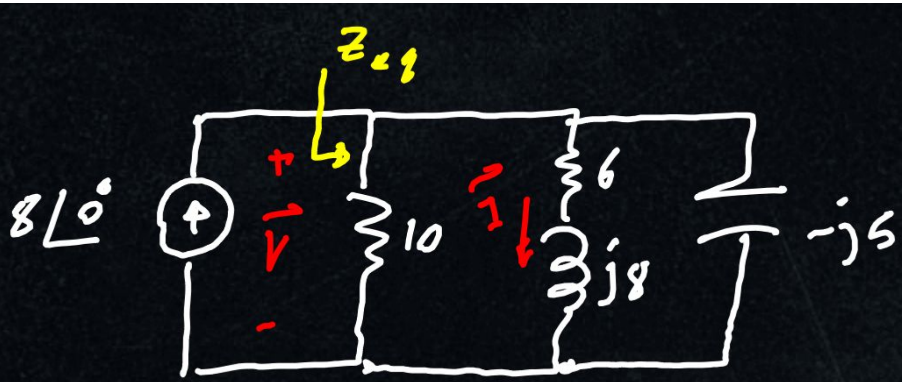
$i_s = 8\cos(20{,}000t)$ A

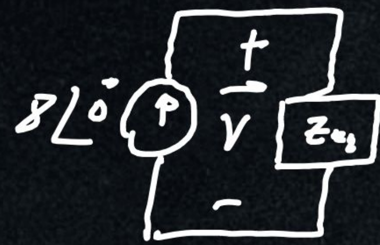


$j\omega L = j(20{,}000)(40 \times 10^{-6})$

$= j8\ \Omega$

$\dfrac{-j}{\omega C} = \dfrac{-j}{20{,}000(10^{-6})} = -j5\ \Omega$



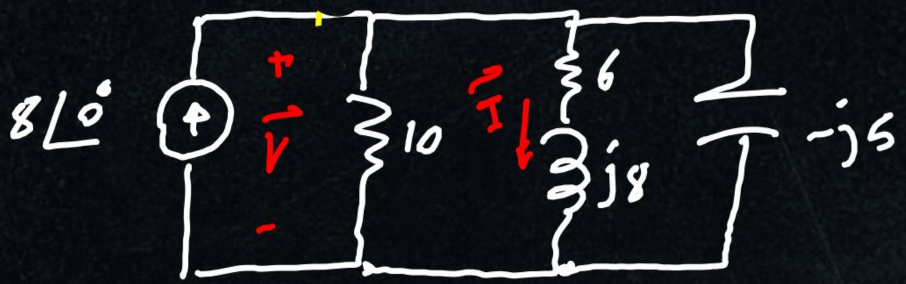


$$Z_{eq} = \frac{1}{\frac{1}{10} + \frac{1}{6+j8} + \frac{1}{-j5}} = \frac{1}{0.1 + \frac{1\angle 0^\circ}{10\angle 53.13^\circ} + j0.2}$$

$$Z_{eq} = \frac{1}{0.1 + 0.1\angle -53.13^\circ + j0.2} = 5\angle -36.87^\circ \ \Omega$$

$$\vec{V} = Z_{eq}\vec{I} = (5\angle -36.87^\circ)(8\angle 0^\circ) = 40\angle -36.87^\circ \ \underline{V}$$

$$\boxed{v = 40\cos(20{,}000t - 36.87^\circ)} \ \underline{V}$$

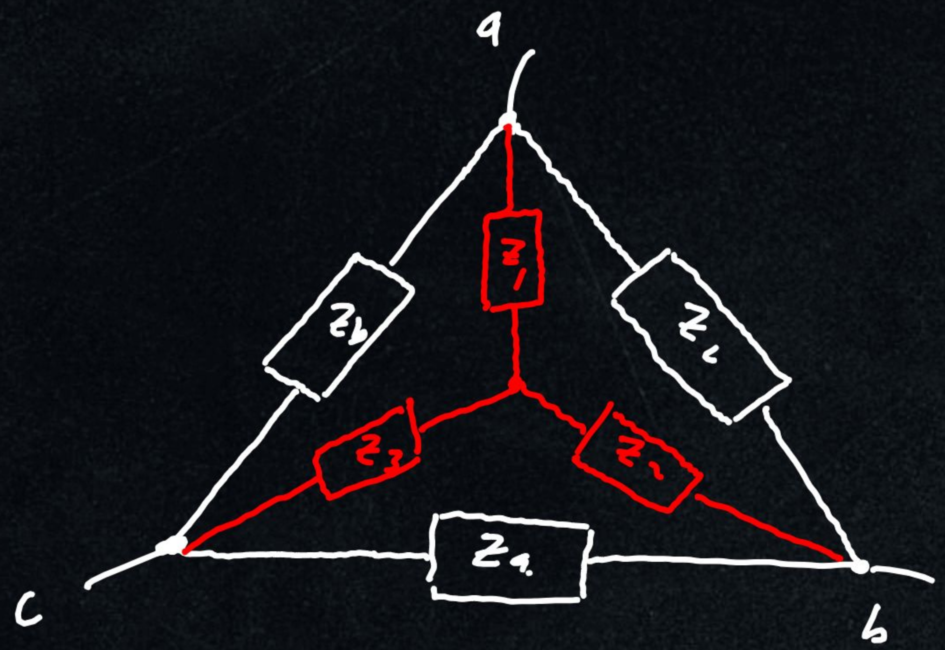


$$\vec{I} = \frac{\vec{V}}{6+j8} = \frac{40\angle -36.87^\circ}{6+j8} = \frac{40\angle -36.87^\circ}{10\angle 53.13^\circ}$$

$$\vec{I} = 4\angle -90^\circ \text{ A}$$

$$i(t) = 4\cos(20{,}000t - 90^\circ) \text{ A}$$

$Y \rightleftarrows \Delta$



$Y \rightarrow \Delta$

$$Z_a = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_2}$$

$$Z_c = (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) / Z_3$$

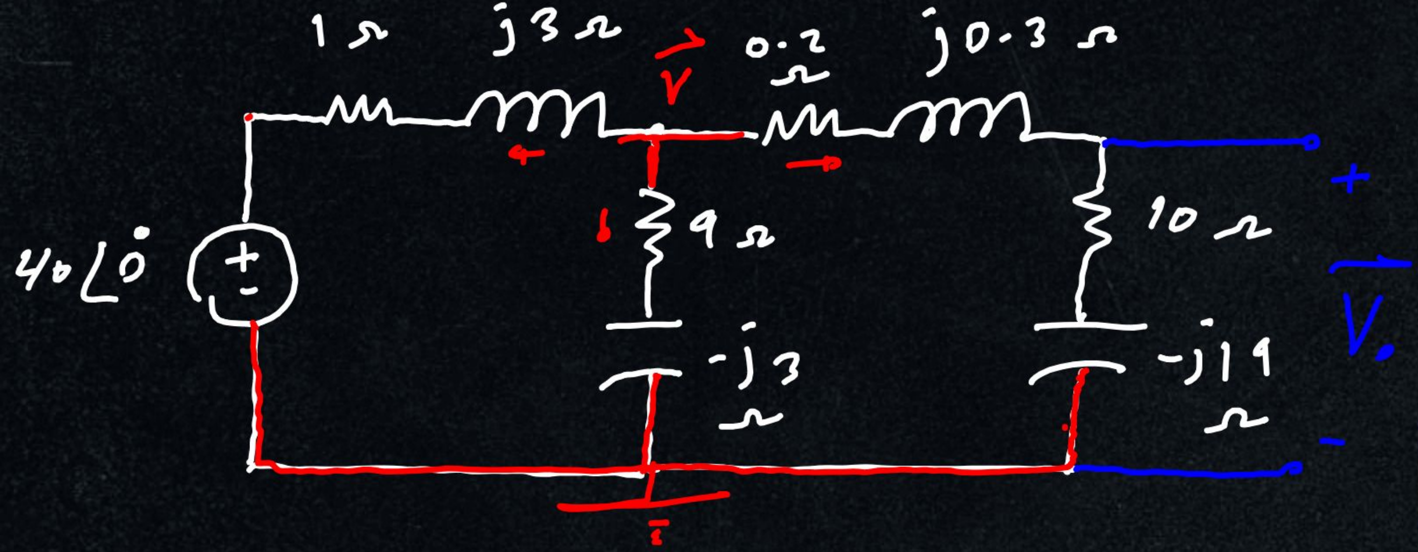
$\Delta \rightarrow Y$

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

EX:- Find $\vec{V}_o$ using Node-Voltage method



Voltage divider

$$\vec{V}_o = \frac{10-j19}{10.2-j18.7}\,\vec{V}$$

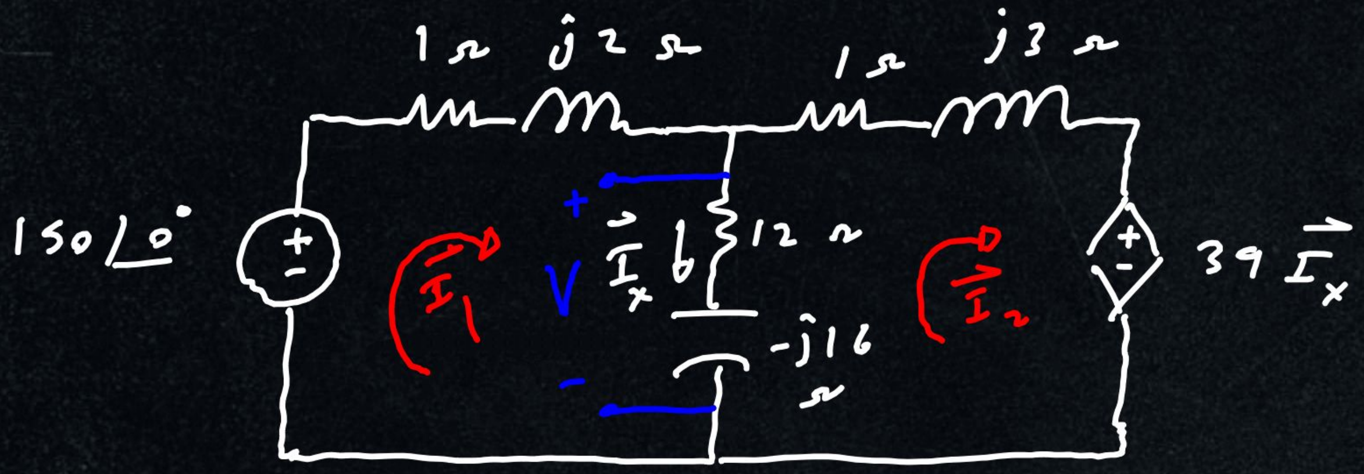
$$\boxed{\vec{V}_o = 36.12 - j18.84}$$

KCL at $\vec{V}$

$$\left(\frac{\vec{V} - 40\angle 0^\circ}{1+j3}\right) + \left(\frac{\vec{V}}{9-j3}\right) + \left(\frac{\vec{V}}{10.2-j18.7}\right) = 0$$

$$\vec{V}\left[\frac{1\angle 0^\circ}{1+j3} + \frac{1\angle 0^\circ}{9-j3} + \frac{1\angle 0^\circ}{10.2-j18.7}\right] = \frac{40\angle 0^\circ}{1+j3} \Rightarrow \text{Find } \vec{V}$$

EX : Find $\vec{V}$ using Mesh-current method :-



$$150\angle 0^\circ = (1+j2)\vec{I}_1 + (12-j16)(\vec{I}_1 - \vec{I}_2) \quad \cdots (1)$$

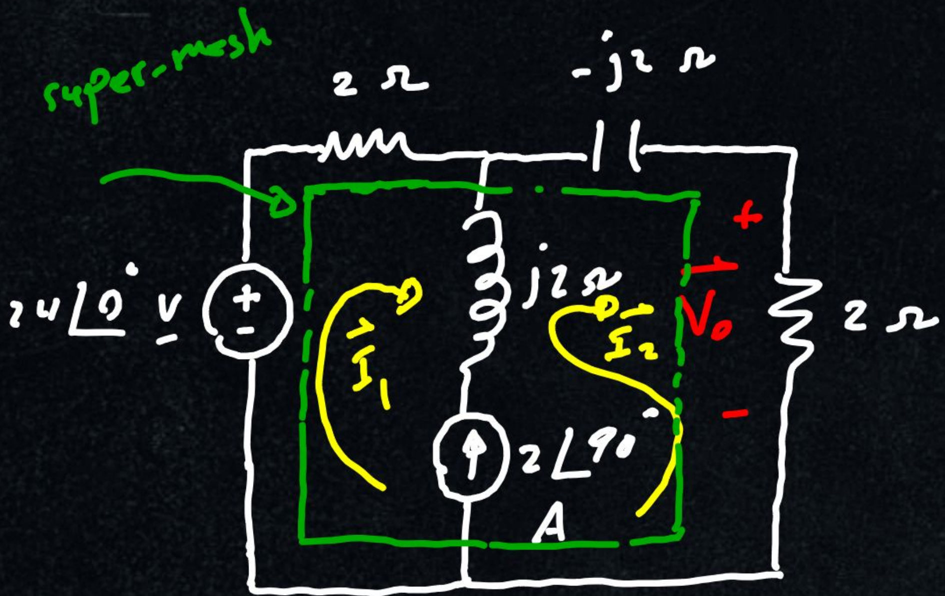
$$(12-j16)(\vec{I}_2 - \vec{I}_1) + (1+j3)\vec{I}_2 + 39\vec{I}_x = 0 \quad \cdots (2)$$

$$\vec{I}_x = \vec{I}_1 - \vec{I}_2 \quad \cdots (3)$$

$\vec{I}_1 = -26 + j52\ A$, $\vec{I}_2 = -24 - j58\ A$, $I_x = -2 + j6\ A$

$$\vec{V} = (12 - j16) I_x = 72 + j104\ V$$

EX :- Use (a) Mesh-current method and (b) Thevenin theorem to find $\vec{V}_o$.



(a) KVL in the super-mesh

$$24\angle 0^\circ = 2\vec{I}_1 + (-j2)\vec{I}_2 + 2\vec{I}_2$$

$$\boxed{24\angle 0^\circ = 2\vec{I}_1 + (2-j2)\vec{I}_2} \quad \text{---} \ (1)$$

$$\boxed{\vec{I}_2 - \vec{I}_1 = 2\angle 90^\circ} \quad \text{---} \ (2)$$

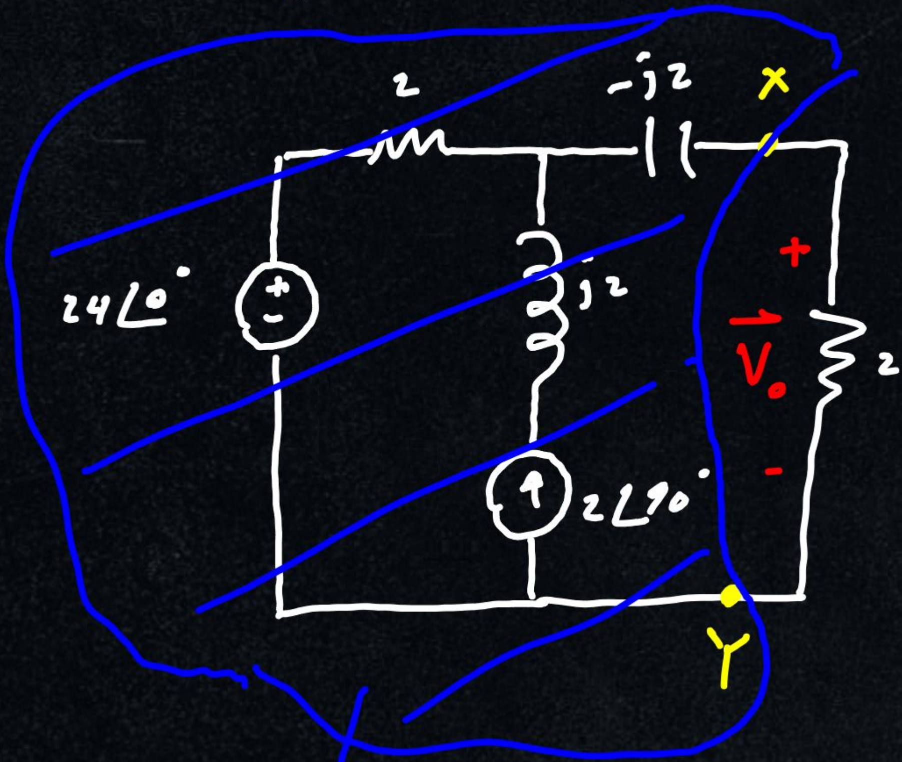
solve for $\vec{I}_2$

(2) → (1)

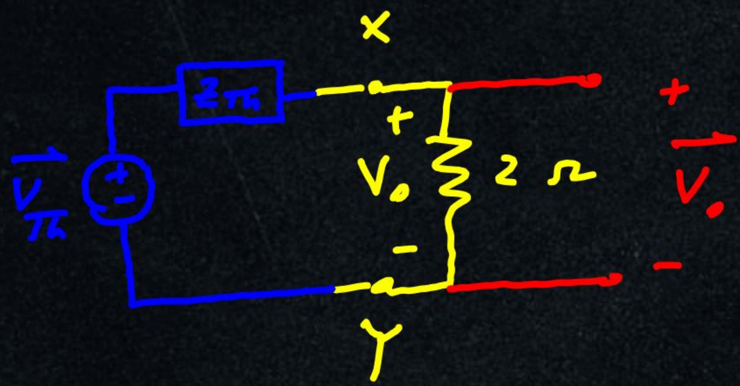
$$\vec{I}_2 = 5.44\angle 36^\circ \ A$$

$$\vec{V}_o = 2\vec{I}_2 = 10.88\angle 36^\circ \ \underline{V}$$

(b)

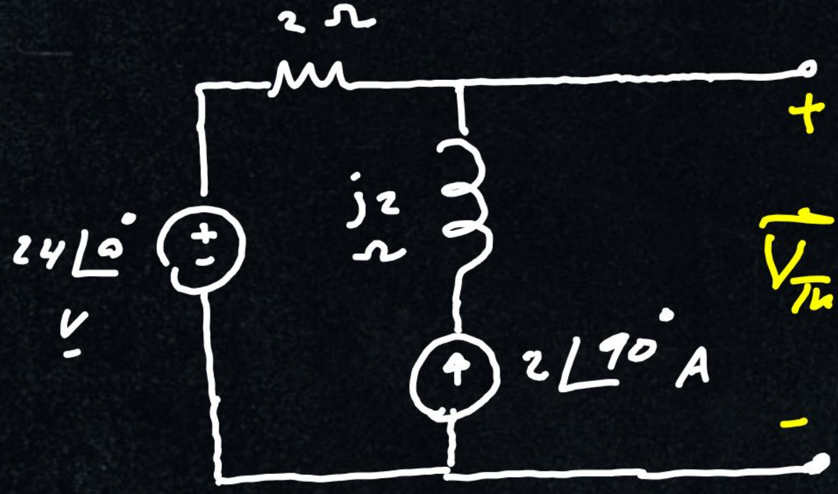


Thevenin Equivalent circuit



$$\vec{V}_o = \frac{2}{2 + Z_{Th}} \cdot \vec{V}_{Th}$$

V_{Th} ??

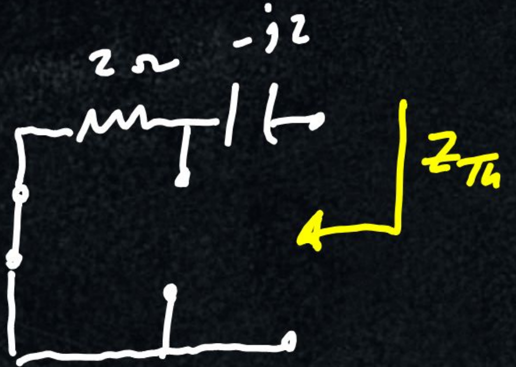


$\vec{V}_{Th} = \vec{V}_{oc}$

$$\vec{V}_{Th} = 2(2\angle 90^\circ) + 24$$

$$\vec{V}_{Th} = 24 + j4 \text{ V}$$

Z_{Th} ?? "Method II"

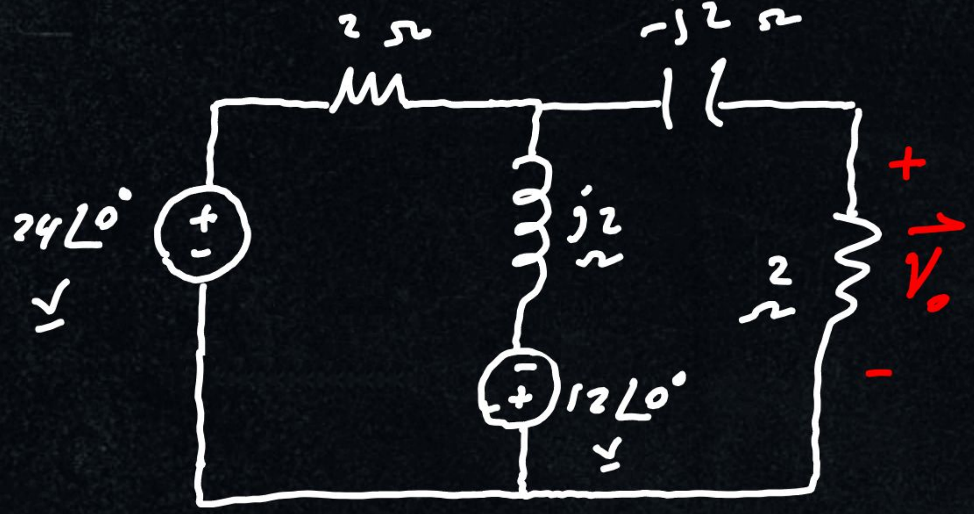


$$Z_{Th} = 2 - j2\ \Omega$$

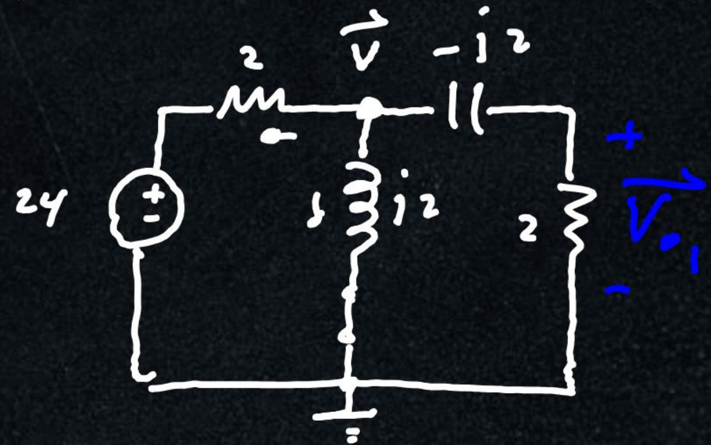
$$\vec{V}_o = \frac{2}{2+Z_{Th}} \cdot \vec{V}_{Th} = \frac{2}{2+(2-j2)}(24+j4)$$

$$\boxed{\vec{V}_o = 10.88\angle 36^\circ \text{ V}}$$

EX:- Find $\vec{V}_o$ using superposition and Thevenin theorem?

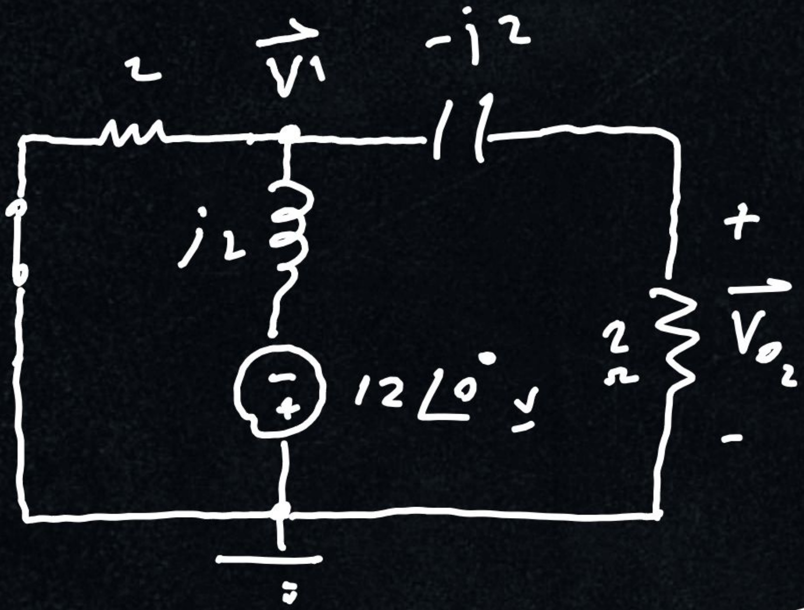


superposition



$$\frac{\vec{V}-24}{2}+\frac{\vec{V}}{j2}+\frac{\vec{V}}{2-j2}=0 \Rightarrow \text{calculate } \vec{V}$$

$$\vec{V}_{o_1}=\frac{2}{2-j2}\vec{V} \Rightarrow \text{calculate } \vec{V}_{o_1}$$



$$\frac{\vec{V}'}{2} + \frac{\vec{V}'+12}{j2} + \frac{\vec{V}'}{2-j2} = 0$$

$\Rightarrow$ calculate $\vec{V}'$

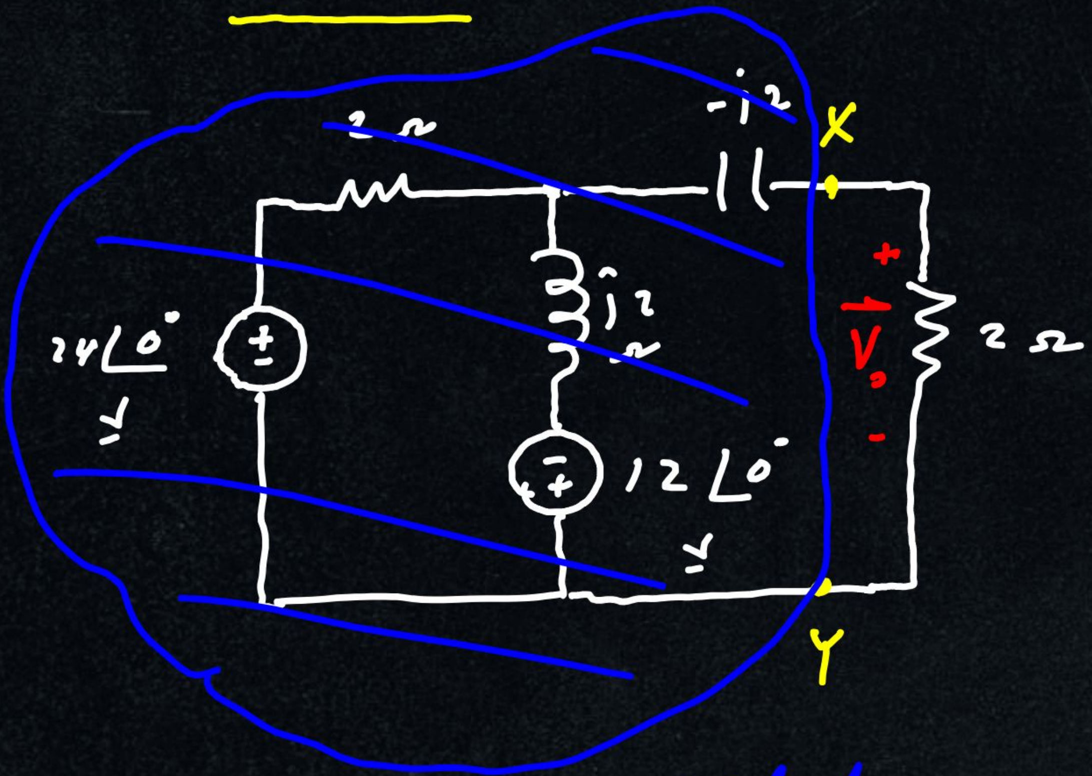
$$\vec{V}_{o_2} = \frac{2}{2-j2}\,\vec{V}'$$

$\Rightarrow$ calculate $\vec{V}_{o_2}$

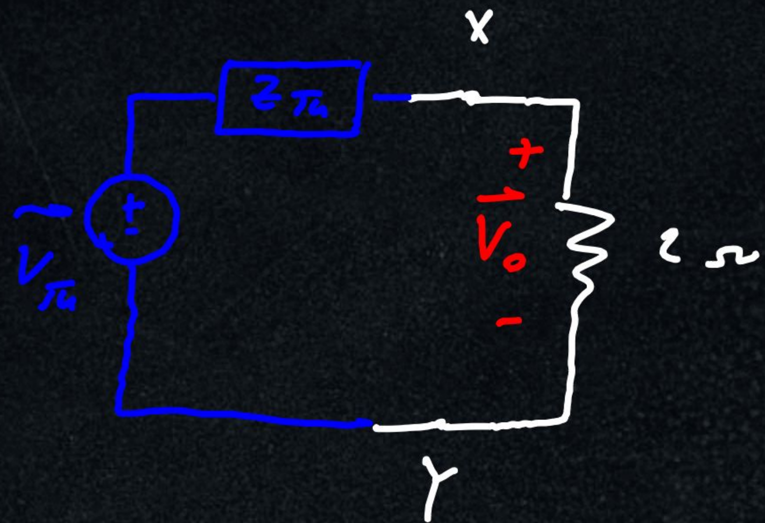
$$\vec{V}_o = \vec{V}_{o_1} + \vec{V}_{o_2} = 12\angle 90^\circ .$$

1

Theverin



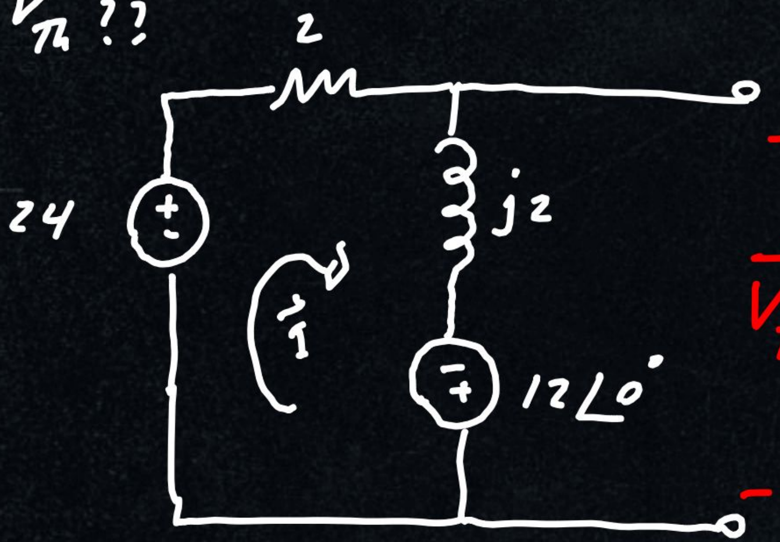
Thevenin equivalent circuit



$$\vec{V}_o = \frac{2}{2 + Z_{Th}} \cdot \vec{V}_{Th}$$

voltage divider

V_{Th} ??



$\vec{V}_{Th} = \vec{V}_{oc}$

KVL in the loop

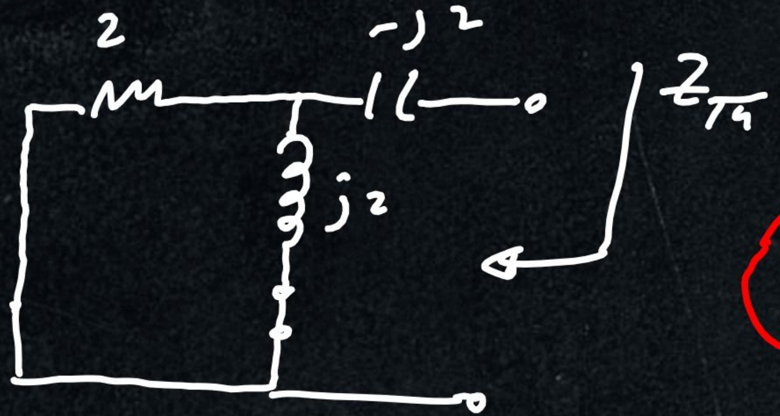
$$-24 + (2+j2)\vec{I} - 12 = 0$$

$$\vec{I} = \frac{36\angle 0°}{2+j2} \text{ A}$$

$$\vec{V}_{Th} = 2(-\vec{I}) + 24$$

$$\vec{V}_{Th} = \frac{72\angle 180°}{2+j2} \text{ V} = \text{—}$$

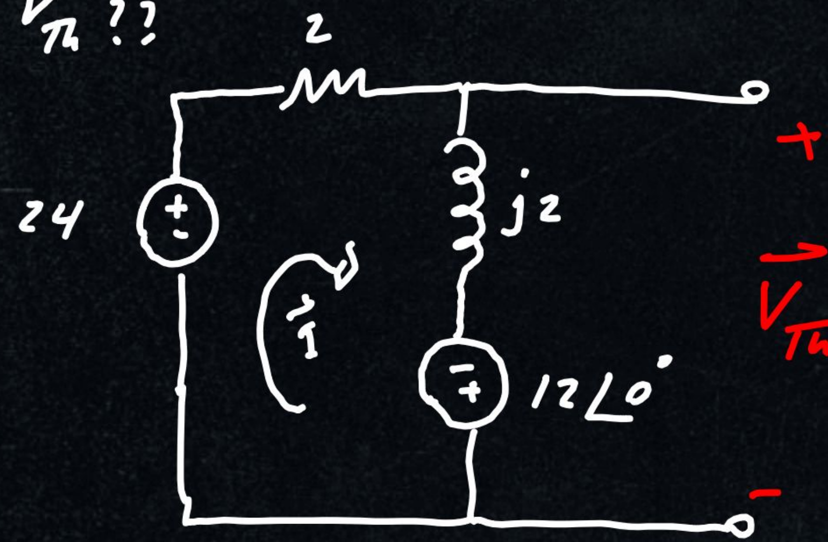
Z_{Th} ?? "Method II"



$$Z_{Th} = (2 // j2) + (-j2) = \text{—}$$

$$\vec{V}_o = \frac{2}{2 + Z_{Th}} \cdot \vec{V}_{Th} = 12\angle 90° \text{ V}$$

V_{Th} ??



$\vec{V}_{Th} = \vec{V}_{oc}$

KVL in the loop

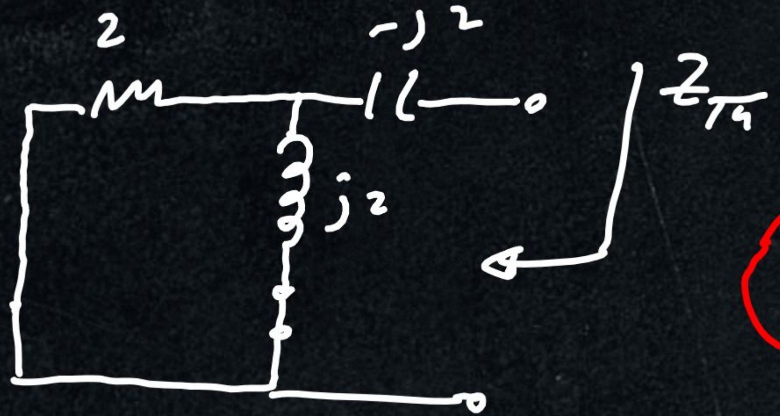
$$-24 + (2+j2)\vec{I} - 12 = 0$$

$$\vec{I} = \frac{36\angle 0^\circ}{2+j2}\ A$$

$$\vec{V}_{Th} = 2(-\vec{I}) + 24$$

$$\vec{V}_{Th} = \frac{72\angle 180^\circ}{2+j2}\ V = \text{—}$$

Z_{Th} ?? "Method II"



$$Z_{Th} = (2 \,//\, j2) + (-j2) = \text{—}$$

$$\boxed{\vec{V}_0 = \frac{2}{2 + Z_{Th}} \cdot \vec{V}_{Th} = 12\angle 90^\circ\ V}$$